INTRODUCING SUPERSYMMETRY

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Abstract:

A systematic and self-contained introduction to supersymmetric model field theories in flat Minkowskian space and to the techniques used in deriving them is given (including superspace). A general overview of supersymmetry and supergravity is provided in the form of an introduction to the main body of the report.

Preface

Modern science shares with both the Greek and earlier philosophies the conviction that the observed universe is founded on simple underlying principles which can be understood and elaborated through disciplined intellectual endeavour. By the Middle Ages, this conviction had, in Christian Europe, become stratified into a system of Natural Philosophy that entirely and consciously ignored the realities of the physical world and based all its insights on thought, and Faith, alone. The break with the medieval tradition occurred when the scientific revolution of the 16th and 17th centuries established an undisputed dominance in the exact sciences of fact over idea, of observation over conjecture, and of

practicality over aesthetics. Experiment and observation were established as the ultimate judge of theory. Modern particle physics, in seeking a single unified theory of all elementary particles and their fundamental interactions, appears to be reaching the limits of this process and finds itself forced, in part and often very reluctantly, to revert for guidelines to the "medieval" principles of symmetry and beauty.

Supersymmetric theories (the subject of this report) are highly symmetric and very beautiful. They are remarkable in that they unify fermions (matter) with bosons (the carriers of force), either in flat space (supersymmetry) or in curved space–time (supergravity). Supergravity naturally unifies the gravitational with other interactions. None of the present model theories is in any sense complete; the hurdles on the way to experimental prediction – and thus to acceptance or rejection – have not yet been cleared. What naive immediate predictions can be made seem to be in disagreement with nature. Yet this particular field of research appears to promise solution of so many outstanding problems that it has excited great enthusiasm in large parts of the theoretical physics community (and equally large scepticism in others). In a truly philosophical spirit it has even been said of the theory that it is "so beautiful it must be true".

It is the purpose of this article to make the concepts, and many of the details, of these theories more widely accessible. I have been fortunate in my friends: many from the "non-particle" domains of the exact sciences, from e.g. astrophysics and chemistry, have expressed an ever more strident wish to be let into the secrets of the theory as it has been developing. They have urged me again and again to exercise all available means of communication to help them understand what it is that we are about. Furthermore, I was pleasantly surprised and greatly encouraged by the vivid interest in an introductory course on the elements of supersymmetric theories which I gave at CERN in the winter of 1981/82. The present article is a widened and, I hope, deepened version of those lectures and of the ones I subsequently gave at Cambridge University and at two Trieste Schools on Supergravity. It is, at present, limited to a thorough discussion of the main aspects of supersymmetric field theories in flat Minkowskian space–time, but I hope to complete a second part on supergravity in the not too distant future.

Whereas, of necessity, the main body of this report is mathematically as rigorous as is compatible with clarity and brevity (I hope to have made this article of use as well to my colleagues in the field as to students or to others wishing to enter the field), a feeling of obligation to a wider community has led me to write, in collaboration with Dr. J.J. Perry*, a rather long and qualitative introduction. In it we attempt to clarify the aims and the structure of the field of supersymmetry and supergravity in such a way that someone who is looking for a general overview may be satisfied by reading only the introduction. It also fits the technical sections into their proper place in the general framework and tries to clarify their raisons d'etre and their interconnections.

1. Introduction

The aim of theoretical physics is to describe as many phenomena as possible by a simple and natural theory. In elementary particle physics, the hope is that we will eventually achieve a unified scheme which combines all particles and all their interactions into one consistent theory. We wish to make further progress on the path which started with Maxwell's unification of magetism and electrostatics, and which has more recently led to unified gauge theories of the weak and of the electromagnetic, and perhaps also of the strong interaction.

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The purpose of this report is to introduce the reader to a development in theoretical particle physics which carries our hopes of being led further along that path: supersymmetry.

Before describing and discussing the origins and the physical content of supersymmetric theories, we give a very brief general definition of such theories. A lengthier discussion of the historical context out of which they have arisen and of their physical and theoretical assumptions follows. Together with a condensed description of the important features of the theories themselves, this forms the main part of this introduction, followed by a summary of where in the text of the report the details are to be found.

Supersymmetry is, by definition, a symmetry between fermions and bosons. A supersymmetric field theoretical model consists of a set of quantum fields and of a Lagrangian for them which exhibit such a symmetry. The Lagrangian determines, through the Action Principle, the equations of motion and hence the dynamical behaviour of the particles. A supersymmetric model which is covariant under general coordinate transformations or, equivalently, a model which possesses local ("gauged") super-symmetry is called a *supergravity* model. Supersymmetric theories describe model worlds of particles, created from the vacuum by the fields, and the interactions between these particles. The supersymmetry manifests itself in the particle spectrum and in stringent relationships between different interaction processes even if these involve particles of different spin and of different statistics.

Both supersymmetry and supergravity aim at a *unified description of fermions and bosons*, and hence of matter and interaction. Supergravity is particularly ambitious in its attempt at *unification of the gravitational with the other interactions*. All supersymmetric models succeed to some degree in these aims, but they fail in actually describing the world as we experience it and thus are models, not theories. We are still striving to find some contact between one of the models and physical reality so that that model could become an underlying theory for nature at its most fundamental level.

By "most fundamental level" particle physicists mean at present the decomposition of matter into quarks and leptons (fermions) and the understanding of all forces between them as arising out of four types of basic interactions, gravitational, weak, electromagnetic and strong. These are described in terms of exchange particles (bosons). The framework within which these building blocks make up a physical theory is relativistic quantum field theory. Seen at this level, "unification" ought to include all four interactions. There is, however, a quantitative and qualitative difference between the gravitational interaction and the others which has had profound consequences both for the structure of the Universe and for our understanding of it.

The purely attractive nature and the long range of the *gravitational force* were responsible for its being the first of the forces in nature to be accessible to the rigorous discipline of experiment and observation. We now know that it is the only fundamental force which adds up for macroscopic bodies and which is independent of the material composition of these bodies. Gravity as a phenomenon is obvious, relatively easy to measure, and – at least in the Newtonian limit – rather simple to understand and to describe. The study of the effects of gravitational forces, first in the solar system, then on massive physical objects in the laboratory, led to the formulation of Newtonian mechanics, the first field in the natural sciences which could be described in purely mathematical terms. The initial understanding of gravity and the subsequent knowledge that grew out of it gave rise to a mathematical description of mechanical processes, based on the concept of force being the cause of accelerated motion. This is still the current understanding of "force". In the Industrial Revolution, this scientific knowledge helped to change the world fundamentally and forever. The "technological forces" of water, steam and machinery, although perhaps not fully understood at their own most fundamental level, could certainly be exploited by a technology that was entirely based on classical mechanics. Technology, in turn, made experiments possible through which physicists learned to keep electric charges apart. They were thus

able to investigate the properties of the *electromagnetic force*, the other long-range force in nature. By the end of the 19th century, this had led to the establishment of the beautiful edifice of classical electrodynamics, culminating in Maxwell's equations and in yet more efficient machines. At the turn of this century, the observation of processes which are basically electromagnetic in nature (the constant speed of light, the spectrum of black-body radiation and the photoelectric effect) brought about a conceptual revolution and with it the breakdown of classical physics. At the same time, the discovery of radioactive decay brought with it the discovery of the short-range *weak* and *strong interactions*; the establishment of quantum mechanics had opened the way for understanding them as well. The understanding of the gravitational interaction took another direction: General Relativity does not see it as an actual "interaction" but rather as a global property of space and time itself. Interestingly enough, both developments played a similarly important role in their impact on the public imagination; the shock of the First World War and of the political revolutions in its wake made the simultaneous breakdown of the old political order manifest, an order which in its sure view of the world had been so akin to the determinism of classical physics.

Unification of gravity with the other forces is an elusive goal. Since gravity is always attractive and long ranging, it would make all complex and large physical objects collapse under their own weight were the other interactions not so much stronger at short distances. This difference in strength, necessary for the world as we see it to exist, has in turn set gravity so far apart from the rest of physics that it is very difficult to think of any experiment which could actually test predictions of a unified theory of all interactions and even less of one that could provide experimental input into the construction of such a theory. The natural domain of Newtonian gravity and of its modern replacement, Einstein's theory of General Relativity, is the world of large distances and massive agglomerations of matter; that of the other forces is the world of atoms, nuclei and elementary particles. Many orders of magnitude separate the two.

The *strong, electromagnetic* and *weak interactions* are now fairly well understood. It has been found that their exchange particles arise naturally in a quantum field theory, if that field theory is required to be locally gauge invariant.

The concept of *gauge invariance* grew out of the observation that if a "charge" (e.g. electric charge, total energy, isospin, etc.) is conserved in a dynamical system, then the Lagrangian for the system is invariant under "global gauge transformations" of the fields. For example, the electric charge is related to invariance under phase transformations $\psi \to e^{iq\theta}\psi$ for all fields ψ which describe particles of charge q. Similarly, the energy is related to time translations $\psi(t, x) \to \psi(t + \Delta t, x)$. The converse is also true (Noether's theorem): if the Lagrangian is invariant under some infinitesimal transformation $\psi \to \psi + \delta\psi$, then there is a conserved current and a conserved charge associated with this gauge invariance. ("Gauge" is an unfortunate misnomer, originating in an attempt by H. Weyl in 1918 to relate the electric charge to a rescaling transformation $\psi \to e^{\lambda}\psi$.) We call the transformations "global" if their parameters do not depend on the space–time coordinates, i.e. if $\theta = \text{const}$. This relationship between conserved quantum numbers and global symmetries of the Lagrangian led, in the 1960's, to a search for globally gauge-invariant field theories capable of describing and classifying all elementary particles. The "8-fold way" was a symmetry very much in this vein and it was in this context that quarks were first postulated as building blocks of strongly interacting matter. But that was not yet the full story.

The requirement of *local gauge invariance* (also called "gauge invariance of the 2nd kind") goes beyond that which can be inferred from charge conservation. We now demand invariance of the Lagrangian under transformations with a space–time dependent parameter $\theta = \theta(x)$. This is in general only possible if an additional field is introduced, the gauge potential, whose quanta will interact with all

the charged particles. This interaction, which results in the exchange of field quanta, will generate forces between the particles. "Gauging" the phase transformations associated with electric charge (i.e. making them x-dependent) forces us to introduce the electromagnetic four-vector potential and, as its quanta, the photons. The result is quantum electrodynamics or, for classical fields, Maxwell's electrodynamics. Requiring other gauge invariances requires additional gauge potentials which give rise to more exchange particles and the other interactions. These exchange particles are the long predicted and recently discovered $W^{\pm}$ and Z^0 for the weak force and the gluons for the strong interactions. The latter have only been indirectly seen in their effects on the distribution of the debris in high-energy particle collisions (jets). To summarise: "gauging" an invariance of the Lagrangian will always give rise to interaction and to forces. Note that the name "gauge theory" is nowadays used exclusively for theories with local gauge invariance.

The gauge transformations under which a Lagrangian is invariant fulfill the axioms for a group (in the mathematical sense): two subsequent ones will again be an invariance, "no transformation" is the unit element, there is exactly one transformation which reverses the effect of each transformation, and three transformations are associative. Using the standard terminology for groups, the respective gauge groups are SU(3) for the strong interactions and SU(2) × U(1) for the electro-weak interactions. The SU(3) transformations act on triplets of quarks whose properties are very similar. They are said to differ only in "colour", hence the name quantum-chromodynamics (QCD) for the SU(3) gauge theory of the strong interactions. The success of gauge theories in describing a variety of elementary particle phenomena in the past ten years has eclipsed the role played by global invariance, and nowadays such global symmetries are thought of as more or less accidental – if indeed they are present. In this context it is already important to mention that local (gauged) supersymmetry will imply gravity.

The theories for both gravitation and elementary particle interactions are well established within their respective domains. In submicroscopic physics, for the masses and distances involved, the deviations introduced by gravity from the flat Minkowskian metric are so minute that elementary particles, or atoms, can safely be treated as if gravitation did not exist. Any "true", i.e., generally covariant, theory should thus be closely approximated by Lorentz-covariant, non-gravitating theories. We must, however, demand of the "true" theory that it be mathematically consistent and that it predict the correct flat limit. Any quantum theory of gravitation so far fails to do so.

Let us argue briefly why the gravitational field ought to be quantised at all. Lacking a better working hypothesis, we assume that the laws of physics extrapolate into extreme regions. The theory of gravitation has so far been tested only in systems where at least one of the masses involved is of astronomical magnitude. Yet this same theory is assumed to also describe the gravitational effects, however weak, between, say, the proton and the electron in an hydrogen atom. If the gravitational field were not quantised, experiments could be imagined which could, in principle, simultaneously measure the position and the momentum of an electron to arbitrary precision. If we decide to postulate the validity of Heisenberg's uncertainty relation even in this extreme case, we must assume the gravitational field to be quantised.

Perhaps we should not extrapolate to very small distances General Relativity's view of space and time as a continuum whose structure is governed by the matter in it. In that case, an entirely different view of the ultimate high energy physics would invalidate most of our present attempts.

Once we assume quantised gravitation, the problem of *renormalisability* becomes very important. Renormalisation is required in quantum field theory in order to make sense of divergent integrals which appear in the perturbation expansion for physical processes. Such expansions are unfortunately the only calculational tools currently available for solving the equations of motion of the theory; they are usually

conceptualised in terms of vaccum polarisation and virtual particle interactions and illustrated by Feynman graphs. In renormalisable theories the divergences which appear can be treated by redefining, in each order of the perturbation expansion, a finite number of theoretical parameters in such a way that the results of "test experiments" are reproduced. Other processes can then be calculated uniquely to the same order. To lowest order, the parameters which must be so renormalised typically represent vacuum energies, masses, coupling constants and factors which multiply wave-functions. Correspondingly, one speaks of "vacuum, mass, coupling constant and wave-function renormalisation". One of the strongest motivations for gauge theories is their renormalisability.

A theory is called non-renormalisable if infinitely many parameters must be redefined. Such a "theory" can make no predictions and is therefore not a theory in the sense of exact science. In general, and this is rather easy to see once the actual mathematical details are considered, coupling constants with negative mass dimensions (for $\hbar = c = 1$) lead to non-renormalisable theories. No matter how we attempt to quantise gravity, we end up with a field theory whose coupling constant, Newton's constant G, has dimension $[1/\text{mass}^2]$ in these units and quantum gravity is therefore non-renormalisable.

The energy at which gravity and quantum effects become of comparable strength can be estimated from the only expression with the dimension of an energy that can be formed from the constants of nature $\hbar$, c and G,

$$E_{\text{Pl}} = c^2 \sqrt{\hbar c / G} \approx 10^{19} \, \text{GeV} \,. \tag{1.1}$$

It is in the region of this energy, the *Planck energy*, where our present theories for the gravitational and the other interactions become incompatible with each other and where we expect a possible unification of the interactions to become manifest. A point particle with Planck mass would have a Schwarzschild radius equal to twice its Compton wavelength. The very remoteness of such an energy region (10^{19} GeV equals about 500 kilowatt-hours) eliminates all hope for direct experiment. Perhaps, if we are very lucky, some isolated prediction of such a unified theory could be testable on a system that integrates a minute effect over a vast range (proton decay experiments are of this type, where large numbers of available protons can make very small decay probabilities measurable). We can, however, not expect experimental physics to give us much reliable guidance into the "Planck region".

The $SU(3) \times SU(2) \times U(1)$ picture of the other forces is not yet a "unified" one. The only property which unites them is that they are each described by a gauge theory. The fact that the direct product structure of $SU(2) \times U(1)$ is "skew" (by the weak mixing angle) against the natural distinction between the weak and the electromagnetic force may suggest some underlying unified scheme which can predict that angle. A lot of work has, over many years, gone into finding a larger gauge group which would describe all three interactions at some high energy. If such a Grand Unification occurred it is known that it must happen at energies of about 10^{15} GeV, only four orders of magnitude less than E_{Pl}. Grand Unified Theories (GUTs) have had some successes (such as the prediction of the mixing angle) and some failures: at the time of writing the measured lower limit for the proton lifetime is growing beyond the point to which one can stretch the simplest GUTs. In any case, even a Grand Unified Theory would at most unify different kinds of interaction (strong and electro-weak) with each other and different kinds of matter (quarks and leptons) with each other. The unification of matter with interaction is not one of the aims of GUTs.

What is it then that points in the direction of supersymmetric theories for a *solution to the unification problem*? Already the most obvious difference between gravity and, say, electrodynamics, namely the absence of negative gravitational charges, can be shown to imply that only a supersymmetric theory can

unify them. As long as we do not dramatically deviate from standard quantum field theory, and we hope that that will not be necessary, a purely attractive force must be carried by a field with even integer spin. The electromagnetic force, on the other hand, which is – of course – not always attractive, is carried by a field with spin one. A number of no-go theorems, about which we will have to learn more later, forbid any direct symmetry transformations between fields of different integer spin and actually leave supersymmetric theories as the only field theoretical models which might achieve unification of all forces of nature. Supersymmetry transformations do not directly relate fields of different integer spin, rather they relate a graviton (the quantum of gravity, with spin 2) to a photon (spin 1) via a spin $\frac{3}{2}$ intermediary, the "gravitino". A partial unification of matter (fermions) with interaction (bosons) thus arises naturally out of the attempt to unite gravity with the other interactions.

The no-go theorems imply that supersymmetry and supergravity are the only possibilities for unification within the framework of quantum field theory. Failure of these theories to ultimately give results which are compatible with the real world would force us to give up either unification or quantum field theory. It is our reluctance to do so which keeps this particular research alive in spite of the absence, as it has been said, of "even a shred of experimental evidence". There is, in principle, one justification for research into theoretical physics other than predicting new data. That is to interpret existing data over a wider range of parameters than before. Therefore, if supergravity could come up with a model which is self-consistent and incorporated General Relativity at the long-distance low-energy end of its phenomenology and "standard" particle physics on the high-energy side, then it would be a grand success even without any "new" testable physics. This latter possibility is by no means ruled out.

Apart from the no-go theorems, there is a further, more technical point that singles out supersymmetric theories: they may resolve the non-renormalisability problem of quantised gravity. In perturbative quantum field theory fermions and bosons tend to contribute with opposite signs to higher-order corrections. Supersymmetry almost always achieves a fine-tuning between these contributions which makes some a-priori present and *divergent terms vanish*. For a long time now there has been great optimism that one of the supergravity models may be entirely free of infinities and thus be a consistent model of quantised gravitation with no need for renormalisation. This hope is slowly fading, but only to the extent that no obvious proof can be found for the conjectured finiteness. What remains is a remarkably friendly behaviour of all supersymmetric theories when it comes to quantum divergences, and the conviction of many theorists that it would be surprising if Nature didn't make use of it. We will later have to say more about *cancellations of divergences* and the enormous interest which they have aroused, not only for gravity but also for the "hierarchy problem" of GUTs, the thirteen orders of magnitude between the GUT mass of 10^{15} GeV/c^2 and the W-boson mass. Normally, a gap of this size is not stable in perturbation theory because of the considerable admixture of the large mass to the small one through exchange of virtual particles and through vacuum polarisation. The gap can only be maintained by repeated fine-tuning up to high orders in the perturbation expansion. In supersymmetric versions of GUTs new particles are exchanged and pair-created, and these new processes cancel some of the effects of the old ones. Mass mixing and consequent fine-tuning can usually be avoided, and the hierarchy, once established, is stabilised.

Progress in physics as in all the sciences has almost always been based on an interplay of two quite different approaches to nature: the one starts by collecting and ordering observational or experimental data ("Tycho"), then describes these data by a small number of empirical laws ("Kepler"), and finally "explains" these laws by a theory, based on a few principles ("Newton"). Theoretical predictions of the outcome of further, more refined, observations and experiments can then be made ("discovery of

Neptune"). The other approach starts from an idea, formulates it in terms of a theory, and proceeds to make predictions which then act as test of the theory and of the original idea. The latter approach – in its pure form – has been most dramatically and singularly successful in Einstein's development of General Relativity. Supersymmetry has also started with just an idea (although without an Einstein), and at the moment we are all searching for the physical evidence to confirm it.

Let us establish some of the rules of the game. For one, the field is still a game, albeit a serious one. First and foremost we postulate the *existence* of a symmetry between fermions and bosons which should underly the laws of physics. No experimental observation has yet revealed particles or forces which manifestly show such a symmetry. Therefore the development of a theory based on supersymmetry requires an understanding not only of how the various symmetry transformations affect each other (the *algebra*) but also of all possible systems (*multiplets* of particles or quantum fields) on which the supersymmetry transformations can act. The symmetry operations will transform different members of a multiplet into each other. More precisely, the transformations are to be represented by linear operators acting on the vector space (the "representation space") spanned by the multiplet. Finally, the theory must predict the time development of interacting physical systems. This is usually achieved by finding appropriate Hamiltonians or *Lagrangians*. The supersymmetry present in the physical system will manifest itself in an invariance of this Lagrangian – or rather of its integral over all time, the action – if all fields undergo their respective supersymmetry transformations. Because of the lack of experimental input, a large fraction of the research effort of supersymmetry theorists has, in fact, been devoted to the finding of, and exploration of, possible supersymmetry respecting interactions.

The theoretical framework in which to construct supersymmetric models in flat space–time is quantum field theory, and it must be pointed out that

> *the standard concepts of quantum field theory allow for supersymmetry without any further assumptions.*

The introduction of supersymmetry is not a revolution in the way one views physics. It is an additional symmetry that an otherwise "normal" field theoretical model can have. As we shall see, all that is required for a field theory to be supersymmetric is that it contains specified types and numbers of fields in interaction with each other and that the various interaction strengths and particle masses have properly related values. As an example, consider the SU(3) gauge theory of gluons, which can be made supersymmetric by including a massless neutral colour octet of spin $\frac{1}{2}$ particles which are their own antiparticles. Jargon has it that such spin $\frac{1}{2}$ partners of the gluons are called "gluinos". If our model contains not only gluons but also quarks, we must also add corresponding partners for them. These have spin 0 and are commonly called "squarks". (Procedures like these are employed particularly in the construction of supersymmetric Grand Unified Theories – *superGUTs or susyGUTs*.)

Before we proceed to discuss the ingredients of supersymmetric models, we must address the question of the Fermi–Bose, matter-force dichotomy. After all, the wave-particle duality of quantum mechanics and the subsequent concept of the "exchange particle" in perturbative quantum field theory seemed to have abolished that distinction for good. The recent triumphal progress of gauge theories has, however, reintroduced it: forces are mediated by gauge potentials, i.e., by vector fields with spin one, whereas matter (including integer-spin mesons) is built from quarks and leptons, i.e., from spin $\frac{1}{2}$ fermions. The Higgs particles, necessary to mediate the needed spontaneous breakdown of some of the gauge invariances (more about this later), play a somewhat intermediate role. They must have zero spin and are thus bosons, but they are not directly related to any of the forces. Purists hope to see them arise as

bound states of the fermions (condensates). Supersymmetric theories, and particularly supergravity theories, "unite" fermions and bosons into multiplets and lift the basic distinction between matter and interaction. The gluinos, for example, are thought of as carriers of the strong force as much as the gluons, except that as fermions they obey an exclusion principle and thus will never conspire to form a coherent, measurable potential. The distinction between forces and matter becomes phenomenological: bosons – and particularly massless ones – manifest themselves as forces because they can build up coherent classical fields; fermions are seen as matter because no two identical ones can occupy the same point in space – an intuitive definition of material existence.

For some time it was thought that symmetries which would naturally relate forces and fermionic matter would be in conflict with field theory. The progress in understanding elementary particles through the SU(3) classification of the "eight-fold way" (a global symmetry) had led to attempts to find a unifying symmetry which would directly relate to each other several of the SU(3) multiplets (baryon octet, decuplet, etc.), even if these had different spins. The failure of attempts to make those "spin symmetries" relativistically covariant led to the formulation of a series of no-go theorems, culminating in 1967 in a paper by Coleman and Mandula [8] which was widely understood to show that it is impossible, within the theoretical framework of relativistic field theory, to unify space–time symmetry with internal symmetries. More precisely, the theorem says that the charge operators whose eigenvalues represent "internal" quantum numbers such as electric charge, isospin, hypercharge, etc., must be translationally and rotationally invariant. This means that these operators commute with the energy, the momentum and the angular momentum operators. Indeed, the only symmetry generators which transform at all under both translations and rotations are those of the Lorentz transformations themselves (rotations and transformations to coordinate systems which move with constant velocity). The generators of internal symmetries cannot relate eigenstates with different eigenvalues m^2 and $l(l+1)\hbar^2$ of the mass and spin operators. This means that irreducible multiplets of symmetry groups cannot contain particles of different mass or of different spin. This no-go theorem, which is discussed in much more detail in section 2, seemed to rule out exactly the sort of unity which was sought. One of the assumptions made in Coleman and Mandula's proof did, however, turn out to be unnecessary: they had admitted only those symmetry transformations which form Lie groups with real parameters. Examples of such symmetries are space rotations with the Euler angles as parameters and the phase transformations of electrodynamics with a real phase angle θ about which we talked earlier. The charge operators associated with such Lie groups of symmetry transformations (their generators) obey well defined *commutation relations* with each other. Perhaps the best-known example is the set of commutators $L_x L_y - L_y L_x \equiv [L_x, L_y] = i\hbar L_z$, for the angular momentum operators which generate spatial rotations.

Different spins in the same multiplet are allowed if one includes symmetry operations whose generators obey *anticommutation relations* of the form $AB + BA \equiv \{A, B\} = C$. This was first proposed in 1971 by Gol'fand and Likhtman [24], and followed up by Volkov and Akulov [70] who arrived at what we now call a non-linear realisation of supersymmetry. Their model was not renormalisable. In 1973, Wess and Zumino [71] presented a renormalisable field theoretical model of a spin $\frac{1}{2}$ particle in interaction with two spin 0 particles where the particles are related by symmetry transformations, and therefore "sit" in the same multiplet. The limitations of the Coleman–Mandula no-go theorem had been circumvented by the introduction of a fermionic symmetry operator which carried spin $\frac{1}{2}$, and thus when acting on a state of spin j resulted in a linear combination of states with spins $j + \frac{1}{2}$ and $j - \frac{1}{2}$. Such operators must and do observe anticommutator relations with each other. They do not generate Lie groups and are therefore not ruled out by the Coleman–Mandula no-go theorem.

In the light of this discovery, Haag, Łopuszański and I [34] extended the results of Coleman and Mandula to include symmetry operations which obey Fermi statistics. We proved that in the context of relativistic field theory the only models which can lead to a solution of the unification problems are supersymmetric theories. The class of such theories is, however, rather restricted: we found that space–time and internal symmetries can only be related to each other by fermionic symmetry operators Q of spin $\frac{1}{2}$ (not $\frac{3}{2}$ or higher) whose properties are either exactly those of the Wess–Zumino model or are at least closely related to them. Only in the presence of supersymmetry can multiplets contain particles of different spin, such as the graviton and the photon. We classified all supersymmetry algebras which can play a role in field theory. These results are presented in more detail in section 2, and we shall here sketch only the essentials of the supersymmetry algebras.

Supersymmetry transformations are generated by quantum operators Q which change fermionic states into bosonic ones and vice versa,

$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,; \qquad Q\,|\text{boson}\rangle = |\text{fermion}\rangle\,. \tag{1.2}$$

Which particular bosons and fermions are related to each other by the operation of some such Q, how many Q's there are, and which properties other than the statistics of the states are changed by that operation depends on the supersymmetric model under study. There are, however, a number of properties which are common to the Q's in all supersymmetric models.

By definition, the Q's change the statistics and hence the spin of the states. Spin is related to behaviour under spatial rotations, and thus supersymmetry is – in some sense – a space–time symmetry. Normally, and particularly so in models of "extended supersymmetry" (which is defined below, $N = 8$ supergravity being one example), the Q's also affect some of the internal quantum numbers of the states. It is this property of combining internal with space–time behaviour that makes supersymmetric field theories interesting in the attempt to unify all fundamental interactions.

As a simple illustration of the non-trivial space–time properties of the Q's consider the following. Because fermions and bosons behave differently under rotations, the Q cannot be invariant under such rotations. We can, for example, apply the unitary operator U which, in Hilbert space, represents a rotation of configuration space by 360° around some axis. Then we get from eq. (1.2) that

$$UQ\,|\text{boson}\rangle = UQU^{-1}U\,|\text{boson}\rangle = U\,|\text{fermion}\rangle$$

$$UQ\,|\text{fermion}\rangle = UQU^{-1}U\,|\text{fermion}\rangle = U\,|\text{boson}\rangle\,.$$

Since fermionic states pick up a minus sign when rotated through 360° and bosonic states do not,

$$U\,|\text{fermion}\rangle = -|\text{fermion}\rangle\,; \qquad U\,|\text{boson}\rangle = |\text{boson}\rangle\,,$$

and since all fermionic and bosonic states, taken together, form a basis in the Hilbert space, we easily see that we must have

$$UQU^{-1} = -Q\,. \tag{1.3}$$

The rotated supersymmetry generator picks up a minus sign, just as a fermionic state does. One can

extend this analysis and show that the behaviour of the Q's under any Lorentz transformation – not only under rotations of the coordinates by $360°$ – is precisely that of a *spinor operator*. More technically speaking, the Q's transform like tensor operators of spin $\frac{1}{2}$ and, in particular, they do not commute with Lorentz transformations. The result of a Lorentz transformation followed by a supersymmetry transformation is different from that when the order of the transformations is reversed.

It is not as easy to illustrate, but it is nevertheless true that, on the other hand, the Q's are *invariant under translations*: it does not matter whether we translate the coordinate system before or after we perform a supersymmetry transformation. In technical terms, this means that we have a vanishing commutator of Q with the energy and momentum operators E and P, which generate space–time translations:

$$[Q, E] = [Q, P] = 0.$$

(1.4)

The structure of a set of symmetry operations is determined by the result of two subsequent operations. For continuous symmetries like space rotations or supersymmetry, this structure is best described by the commutators of the generators, such as the ones given above for the angular momentum operators. The commutator structure of the Q's with themselves can best be examined if they are viewed as products of operators which annihilate fermions and create bosons in their stead, or vice versa. It can then be shown (see section 2) that the canonical quantisation rules for creation and annihilation operators of particles (and in particular the *anti*commutator rules for fermions, which reflect Pauli's exclusion principle) lead to the result that it is the anticommutator of two Q's, not the commutator, which is again a symmetry generator, albeit one of bosonic nature.

Let us consider the anticommutator of some Q with its Hermitian adjoint $Q^\dagger$. As spinor components, the Q's are in general not Hermitian, but $\{Q, Q^\dagger\} \equiv QQ^\dagger + Q^\dagger Q$ is a Hermitian operator with positive definite eigenvalues:

$$\langle \dots | QQ^\dagger | \dots \rangle + \langle \dots | Q^\dagger Q | \dots \rangle = \left| Q^\dagger | \dots \rangle \right|^2 + \left| Q | \dots \rangle \right|^2 \geq 0.$$

(1.5)

This can only be zero for all states $| \dots \rangle$ if $Q = 0$. A more detailed investigation will show that $\{Q, Q^\dagger\}$ must be a linear combination of the energy and momentum operators:

$$\{Q, Q^\dagger\} = \alpha E + \beta P.$$

(1.6)

This relationship between the anticommutator of two generators of supersymmetry transformations on the one hand and the generators of *space–time translations* (namely energy and momentum) on the other, is central to the entire field of supersymmetry and supergravity. It means that the subsequent operation of two finite supersymmetry transformations will induce a translation in space and time of the states on which they operate.

There is a further important consequence of the form of eq. (1.6). When summing this equation over all supersymmetry generators, we find that the βP terms cancel while the αE terms add up, so that

$$\sum_{\text{all } Q} \{Q, Q^\dagger\} \propto E.$$

(1.7)

Depending on the sign of the proportionality factor, the spectrum for the energy would have to be

either ≥ 0 or ≤ 0 because of the inequality (1.5). For a physically sensible theory with energies bounded from below but not from above, the proportionality factor will therefore be positive.

The equations (1.4) to (1.7) are crucial properties of the supersymmetry generators, and many of the most important features of supersymmetric theories, whether in flat or curved space–time, can be derived from them.

One such feature is the *positivity of energy*, as can be seen from eq. (1.7) in conjunction with (1.5):

> *the spectrum of the energy operator E (the Hamiltonian) in a theory with supersymmetry contains no negative eigenvalues.*

We denote the state (or the family of states) with the lowest energy by $|0\rangle$ and call it the *vacuum*. The vacuum will have zero energy

$$E|0\rangle = 0 \tag{1.8a}$$

if and only if

$$Q|0\rangle = 0 \quad \text{and} \quad Q^\dagger|0\rangle = 0 \quad \text{for all } Q. \tag{1.8b}$$

Any state whose energy is not zero, e.g. any one-particle state, cannot be invariant under supersymmetry transformations. This means that there must be one (or more) superpartner state $Q|1\rangle$ or $Q^\dagger|1\rangle$ for every one-particle state $|1\rangle$. The spin of these partner states will be different by $\frac{1}{2}$ from that of $|1\rangle$. Thus

> *each supermultiplet must contain at least one boson and one fermion whose spins differ by $\frac{1}{2}$.*

A supermultiplet is a set of quantum states (or, in a different context, of quantum fields) which can be transformed into one another by one or more supersymmetry transformations. This is exactly analogous to the concept of "multiplet" known from atomic, nuclear and elementary particle physics where, e.g., the proton and the neutron form an iso-spin doublet and can be transformed into each other by an iso-spin rotation.

The translational invariance of Q, expressed by eq. (1.4), implies that Q does not change energy and momentum and that therefore

> *all states in a multiplet of unbroken supersymmetry have the same mass.*

Experiments do not show elementary particles to be accompanied by superpartners with different spin but identical mass. Thus, if supersymmetry is fundamental to nature, it can only be realised as a spontaneously broken symmetry.

The term "spontaneously broken symmetry" is used when the interaction potentials in a theory, and therefore the basic dynamics, are symmetric but the state with lowest energy, the ground state or vacuum, is not. If a generator of such a symmetry acts on the vacuum the result will not be zero. Perhaps the most familiar example of a spontaneously broken symmetry is the occurrence of ferro-magnetism in spite of the spherical symmetry of the laws of electrodynamics. (S. Coleman, in a series of lectures in 1973, used instead the term "secret symmetry". For elementary particle physics, this would

be a better term since the impression is avoided that the "breaking" is in all cases a physical process – a phase transition – which takes place "spontaneously" at some time and which could perhaps be observed. Furthermore, the "secret" emphasises the wish of researchers to find such underlying symmetry.) Because the dynamics retain the essential symmetry of the theory, states with very high energy tend to lose the memory of the asymmetry of the ground state and the "spontaneously broken symmetry" "gets re-established". High energy may mean high temperature, but we do not yet fully understand the behaviour of supersymmetry at non-zero temperatures, essentially because the statistics of the occupation of states is different for fermions and bosons.

If supersymmetry is spontaneously broken, the ground state will not be invariant under all supersymmetry operations: $Q|0\rangle \neq 0$ or $Q^\dagger|0\rangle \neq 0$ for some Q. From what was said above in eqs. (1.8), we conclude that

> *supersymmetry is spontaneously broken if and only if the energy of the lowest lying state (the vacuum) is not exactly zero.*

Whereas spontaneous supersymmetry breaking may lift the mass degeneracy of the supermultiplets by giving different masses to different members of the multiplets, the multiplet structure itself will remain intact. In particular, we still need "superpartners" for all known elementary particles, although these may now be very heavy or otherwise experimentally unobtainable. The superpartners carry a new quantum number (called R-charge). It has been shown that the highly desirable property of superGUTs models, mentioned earlier, namely that they stabilise the GUT hierarchy, is closely associated with a strict conservation law for this quantum number. If nature works that way, the lightest particle with non-zero R-charge must be stable. Whereas this particle may be so weakly interacting that it has not as yet been observed, its presence in the Universe could crucially and measurably influence cosmology.

As a matter of convention (and of amusement), fermionic superpartners of known bosons are denoted by the suffix -ino (hence "gravitino, photino, gluino"); the bosonic superpartners of fermions are denoted by a prefixed s- ("squark, slepton"). The discovery of any such *bosinos* or *sfermions* would confirm the important prediction of superpartners which is common to all supersymmetric models. It would be a major breakthrough and would establish supersymmetry as an important property of the physics of nature rather than just as an attractive hypothesis.

We have not yet specified "how much" supersymmetry there should be. Do we propose one spin $\frac{1}{2}$ photino as a partner for the physical photon, or two, or how many? Different supersymmetric models give different answers, depending on how many supersymmetry generators Q are present, as conserved charges, in the model. As already said, the Q are spinor operators, and a spinor in four space–time dimensions must have at least four real components. The total number of Q's must therefore be a multiple of four. A theory with minimal supersymmetry would be invariant under the transformations generated by just the four independent components of a single spinor operator Q_α with $\alpha = 1, \ldots, 4$. We call this a

> *theory with $N = 1$ supersymmetry,*

and it would give rise to, e.g., a single uncharged massless spin $\frac{1}{2}$ photino which is its own antiparticle (a "Majorana neutrino"). If there is more supersymmetry, then there will be several spinor generators with four components each, $Q_{\alpha i}$ with $i = 1, \ldots, N$, and we speak of a

> *theory with N-extended supersymmetry,*

which will give rise to N photinos. The fundamental relationship (1.6) between the generators of supersymmetry is now replaced by

$$\{Q_i, (Q_j)^\dagger\} = \delta_{ij}(\alpha E + \beta P).$$ (1.9)

Most models with extended supersymmetry are naturally invariant under rotations of their -inos into each other. These rotations form an internal symmetry group (somewhat like iso-spin). In the case of supergravity, this invariance can be "gauged", i.e., made local, and we arrive at a natural link between space–time symmetries (general coordinate invariance and supersymmetry) and gauge inter-actions. This exciting possibility speaks very much in favour of extended supergravities.

On the other hand, extended supersymmetry is not without serious drawbacks. An important argument against its physical relevance is that it does not allow for chiral fermions as they are observed in nature (neutrinos). The reasons for this can only be understood after a more detailed study of supersymmetric gauge theories, and we shall come back to this point in section 12. Here it must suffice to say that in order to retain the stabilising influence of $N = 1$ supersymmetry on the GUT mass hierarchy on the one hand, and to permit chiral fermions on the other, we must have a breakdown chain of the type

$$\begin{matrix} \text{extended supersymmetry} & \xrightarrow{} & N = 1 \text{ supersymmetry} & \xrightarrow{} & \text{no supersymmetry} \\ \text{(at large } E\text{)} & & \text{(at medium } E\text{)} & & \text{(at low } E\text{)} \end{matrix}$$

with decreasing energy. The final breakdown of all supersymmetry in the last step is necessary to lift the mass degeneracy of the supermultiplets. For spontaneous supersymmetry breaking at least, there is a serious problem with this picture: for extended supersymmetry, eq. (1.7) remains valid even if the sum runs only over the space–time indices on $Q_{\alpha i}$,

$$\sum_{\alpha=1}^{4} \{Q_{\alpha i}, (Q_{\alpha i})^\dagger\} \propto E \qquad \text{for each } i.$$ (1.10)

Earlier, we saw that this relationship meant that a non-zero vacuum expectation value of E was a necessary and sufficient condition for spontaneous supersymmetry breaking, i.e. for $Q|0\rangle \neq 0$. This is now the case for each of the individual N supersymmetries, and we conclude that either none of them is spontaneously broken, or that all of them are, depending on whether $\langle E\rangle = 0$ or not. Furthermore, the proportionality factor in (1.10) is the same for all values of i and we therefore expect the effects of supersymmetry breaking to set in at the same energy for all Q_i. This makes a breakdown hierarchy such as the one required seem impossible.

Arguments such as the one just presented hold strictly only in the absence of gravity. In the context of supergravity, it may therefore be possible to overcome these difficulties and to arrive at a scenario of hierarchical supersymmetry breaking after all. This has indeed been achieved for one of the Kaluza–Klein supergravities. Kaluza–Klein theories [12] are characterised by additional spatial dimensions in which the space is very highly curved (radii in the region of the Planck length). Deviations from the phenomenology of flat space are therefore particularly large for such models, and it should not surprise us to see many "no-go theorems" overcome. It is typical of supersymmetry that a problem encountered at one end of the spectrum of physical research, such as the absence of right-handed neutrinos, should lead us to search for a solution at another end, such as in the structure and

dimensionality of space–time itself. It is successes like this which make the field of supersymmetry and supergravity so exciting and potentially so rewarding.

Is there a maximal number N_{max} of supersymmetries that a field theory or a supergravity model can have? The answer is yes, and the limits originate in the requirement that the symmetry transformations should act on multiplets of physical states and that the underlying theory must be either General Relativity or a renormalisable field theory in flat space. From the point of view of the algebra alone, as summarised in eqs. (1.4) and (1.10) and as described in section 2, there would be no limits on N. Let us try to understand how the limits actually arise and what they are.

Consider a particle such as the photon or the graviton. If we apply a supersymmetry transformation to that particle, we get an -ino:

photon ↔ photino .

The same supertransformation applied a second time will give us the original particle back, but translated to a different point in space and time – this is the essence of eq. (1.6).
If we have additional, different types of supersymmetry transformations, as in the case of $N > 1$, then another type will produce a different -ino. If applied twice, the new supersymmetry transformation will give back the original particle in a displaced position, just as the old one did. But what happens if we first apply one type of transformation and then the other? We cannot get back the original photon, because of the Kronecker δ-symbol in eq. (1.9). Instead, we get yet other superpartners with new properties,

photon ⟷ first photino ⟷ more superpartners ,
⟷ second photino ⟷

and a detailed study (section 3) will show that they normally have spins different from both the original particle and from the -inos. There is a minimal range of spins covered by multiplets, and this range increases with N. More quantitatively,

any multiplet of N-extended supersymmetry will contain particles with spins at least as large as $\frac{1}{4}N$.

This means that spins $\geq \frac{3}{2}$ must be present for $N > 4$ and spins $\geq \frac{5}{2}$ for $N > 8$. Renormalisable flat-space field theories cannot accommodate spins $\geq \frac{3}{2}$. (The canonical dimensions of fields which describe particles increase with spin, and at a certain threshold, which happens to be at spin $\frac{3}{2}$, their presence requires the introduction of coupling constants with negative mass dimensions. These make field theories non-renormalisable.) In addition, since gravity cannot consistently couple to spins $\geq \frac{5}{2}$, we arrive at the famous limits

$N_{max} = 4$ *for flat-space renormalisable field theory* ,

$N_{max} = 8$ *for supergravity* .

Both *maximally extended supersymmetric theories*, namely $N = 4$ super-Yang–Mills theory and $N = 8$ supergravity, *are essentially unique*: there is only one multiplet in each case whose spins fit into the slots $0 \leq s \leq 1$ and $0 \leq s \leq 2$, respectively. These models have been studied extensively, and it has been

shown that the $N = 4$ theory is not only renormalisable but actually finite. The infinite contributions to perturbation theory from the different types of particles in the model cancel exactly. This is discussed in slightly more detail in the section on $N = 4$ supersymmetry. For a model field theory, this property is highly remarkable and possibly of importance for the rigorous discipline of research into quantum field theory. For physics, unfortunately, the $N = 4$ theory seems irrelevant, since its spectrum (which is uniquely determined) in no way resembles that of the world of elementary particles.

$N = 8$ supergravity, and particularly the version which has a local, "gauged" O(8) invariance which rotates the gravitinos into each other, has been hailed as the best candidate to date for a Theory of Everything. Like all models with extended supersymmetry, it can be understood as derived from a field theory in more than four dimensions (more about this in section 14). The $N = 8$ model, at least in its non-gauged version, can be derived from a relatively simple model in eleven space–time dimensions. It actually provides a rather natural separation between the four dimensions of the "real world" and the other seven, which get spontaneously compactified. It remains to be seen whether $N = 8$ supergravity has other, yet more miraculous properties which could pave the way for the unification of gravity with quantum mechanics. Unfortunately, it does not seem to share the property of finiteness with the $N = 4$ theory.

The present report treats flat-space supersymmetry. It is meant to lay a base for work in the field of superGUTs as well as to be a stepping stone towards supergravity, the subject of a subsequent report. I intend to hand over to the reader a tool kit of techniques and sound basic knowledge of the field which should enable him or her to thoroughly enjoy the vast amount of literature on the subject. In order to achieve this, I have tried to make almost every step of reasoning and of algebra as transparent as possible, only relying on the "it can be shown that . . ." when I felt that it can indeed be shown using the techniques presented in this report. Whenever that is not so, I say it (like in the case of superspace perturbation calculations).

In section 2, a detailed derivation of the supersymmetry algebras is presented. Section 3 describes properties of their representations on multiplets and includes the derivation, step by step and in full detail, of the simplest field multiplet. Further field multiplets, together with the rules of their tensor calculus, are introduced in section 4. In section 5 the first supersymmetric model field theory is described, the Wess–Zumino model, which is used as an example to illustrate the appearance of auxiliary fields and the problem of on-shell/off-shell representations. Also in section 5, the supercurrent makes its first appearance.

After a diversion into a form of spontaneous supersymmetry breaking which does not involve gauge theories (section 6), the concept of superspace is introduced. Sections 7 and 8 are devoted entirely to superspace and superfield techniques. Nowadays, both the literature on supersymmetry and the language of those working in the field uses a mixed nomenclature in which things are explained sometimes in component (multiplet) language and sometimes in terms of superfields, whichever is the more convenient for the task at hand. In the later sections of this report, I also use the same mixture of language. Therefore, in sections 7 and 8, I try to clarify this somewhat confusing habit by giving careful treatment of the correspondences between superfields and multiplets.

Sections 9 and 10 deal with the very important subject of supersymmetric gauge theories. With an eye on supergravity, I chose two approaches and presented them both. First, in section 9, I introduce super-QED as the theory generated by coupling radiation to the multiplet which contains the electromagnetic current ("Noether coupling"). Using the multiplet of the energy-momentum tensor instead, one could come to grips with supergravity in a similar way. Subsequently, in section 10, the complementary approach is developed where supersymmetric gauge theories are introduced

geometrically, starting from superspace. It turns out that crucial input has to come from elsewhere ("the constraints"). This input could be the results of the Noether procedure, or it could be some other knowledge such as that of the structure of those multiplets which are supposed to exist in the geometric background created by the gauge theory. The same will be true for supergravity, which can be derived using a similar tandem approach, involving geometry and Noether coupling to the multiplet which contains the supercurrent and the energy-momentum tensor.

More about spontaneous supersymmetry breaking is said in section 11, this time in the presence of gauge interactions.

Sections 12 to 14 introduce extended supersymmetry. It is shown how one can arrive at $N = 2$ and $N = 4$ by using special cases of the general $N = 1$ theory. The finiteness properties are discussed. Section 14 deals with dimensions higher than four. This is meant to provide a basis for the reader who wishes to go on to Kaluza–Klein or string theories, or to the many other aspects of extended supersymmetry and supergravity which are often described in more than four dimensions. A thorough introduction into the properties of spinors and Dirac matrices for arbitrary space–times is provided in section 14 in conjunction with appendix A.7.

Section 15 is devoted to the supercurrent and the properties of the multiplet in which it sits. This section provides the natural link to supergravity, since the energy-momentum tensor must be contained in the multiplet of the supercurrent. Since the energy density gives rise to the gravitational field, the super-charge density must give rise to a particle field in the same multiplet with the graviton. This is the gravitino which has spin $\frac{3}{2}$. In this picture of Noether couplings where currents are the sources for particle fields, section 15 will be to supergravity what subsection 9.1 on the electromagnetic current superfield was to super-QED.

Finally, in section 16, conformal supersymmetry is introduced as a natural consequence of the structure of the supercurrent multiplet. Conformal supergravity and its tensor calculus are useful tools for the construction of supergravity representations. It may also very well be that the unbroken conformal invariance of the $N = 4$ theory is highly significant for field theory.

The Technical Appendix contains formulas and notation in somewhat more depth than usual. Particularly subsections A.2–3 and A.7 on finite dimensional representations of groups and on the Dirac matrices for $O(d_+, d_-)$ could almost be independent sections. Having them in the Appendix does, however, make them easier to find!

A list of references concludes the article, and I wish to conclude this introduction with a few words on referencing.

This report covers the basics of supersymmetry and its techniques. Much of the matter presented here was well developed within a few years of the appearance of Wess and Zumino's first paper and of Salam and Strathdee's paper on superspace. I have chosen my presentation with the hindsight of the many years of research that have passed since, and with the knowledge of the existence and the shape of supergravity theories. Nevertheless, the material itself is rather old, and many of my colleagues, some with names which have become household words among supersymmetrists, will look in vain for references to their later work. I apologise where this was an oversight, and I ask for understanding where I simply could not, in this context, cover the results for which they are famous. This holds in particular for all the work on supergravity.

In previous Physics Reports [82, 83] and books [26, 80] on the subject, the authors have either chosen to include every reference in existence at the time, or they have restricted references to a symbolic number such as two per chapter or none at all. In each case any argument about proper referencing could easily be defused. Since it would clearly be inappropriate to give no references in a

report like this one, and since the total number of published papers on the subject had exceeded 1500 already in summer 1982 when I compiled a computer list, neither of these two elegant ways out of controversy were available to me. I chose instead to give a relatively small number of references where I felt that they were necessary, and I hope not to have offended too many old friends, nor to have made too many new enemies.

2. Generators of supersymmetry and their algebra

In this section, Coleman and Mandula's theorem about the generators of symmetries [8] is described, and those additional possibilities are discussed which arise if statistics-changing symmetries (supersymmetries) are admitted. We shall see that supersymmetry generators necessarily must obey anticommutator relations with each other. This section is based mainly on the 1975 paper by Haag, Łopuszański and myself [34].

2.1. Generators of symmetries

The generator of a symmetry is an operator in Hilbert space that replaces one incoming or outgoing multiparticle state with another and furthermore "leaves the physics unchanged". Such an operator should act additively on direct products of states, and this implies that it can be written as the product of an annihilation operator a, which picks a particle out of a multiparticle state and annihilates it, and a creation operator $a^\dagger$, which creates another particle in its stead with different properties and different 3-momentum. The most general such product is

$$G = \sum_{ij} \int d^3p \, d^3q \, a_i^\dagger(p) \, K_{ij}(p, q) \, a_j(q).$$ (2.1)

This operator is determined completely by the integral kernel $K_{ij}(p, q)$, a c-number function of the momenta p and q and of the other particle properties, denoted by the indices i and j. Using the symbol $*$ for the convolutions in (2.1), we can write G symbolically as

$$G = a^\dagger * K * a.$$ (2.1a)

The operator G will replace some incoming quantum state $|\text{in}\rangle$ by another, $G|\text{in}\rangle$. G is called a *generator of a symmetry* if, in addition, it commutes with the S-matrix, i.e., if it does not matter whether we "reshuffle" the state before or after an interaction has taken place: $SG|\text{in}\rangle = GS|\text{in}\rangle$ or

$$[S, G] = 0.$$ (2.2)

This is a formal expression for what was called "leaving the physics unchanged".

The sums in (2.1) run over all particle quantum numbers and thus also over all spin values. There are terms which replace bosons by bosons, and others which replace fermions by fermions, bosons by fermions and fermions by bosons. Any G can be decomposed into an *even part B* and an *odd part F*,

$$G = B + F,$$ (2.3)

where B contains only terms which replace bosons by bosons and fermions by fermions, while F replaces bosons by fermions and fermions by bosons. Symbolically, we can write

$$B = b^\dagger * K_{bb} * b + f^\dagger * K_{ff} * f$$

$$F = f^\dagger * K_{fb} * b + b^\dagger * K_{bf} * f,$$

(2.4)

where b and f are annihilation operators for bosons and fermions, respectively. We assume validity of the spin-statistics theorem and therefore the B's may change spin by integer amounts or not at all, whereas the F's must change the total spin of a state by a half-odd amount, and are thus necessarily *super*symmetry generators.

2.2. Canonical quantisation rules

We assume that the particle operators obey canonical quantisation rules:

$$[b_i(\boldsymbol{p}), b_j^\dagger(\boldsymbol{q})] = \delta_{ij}\,\delta^3(\boldsymbol{p} - \boldsymbol{q}); \qquad [b, b] = [b^\dagger, b^\dagger] = 0$$

(2.5a)

$$\{f_i(\boldsymbol{p}), f_j^\dagger(\boldsymbol{q})\} = \delta_{ij}\,\delta^3(\boldsymbol{p} - \boldsymbol{q}); \qquad \{f, f\} = \{f^\dagger, f^\dagger\} = 0$$

(2.5b)

$$[b, f] = [b, f^\dagger] = [b^\dagger, f] = [b^\dagger, f^\dagger] = 0.$$

(2.5c)

These relations can be written most elegantly: we observe that $\delta_{ij}\,\delta^3(\boldsymbol{p} - \boldsymbol{q})$ is the unit element $\mathbb{1}$ of the convolution product $*$, and we introduce the "graded commutator" symbol $[.., ..\}$ which denotes the anticommutator if both operators are fermionic and the commutator in all other cases. The canonical quantisation rules then read simply

$$[a, a^\dagger\} = \mathbb{1}; \qquad [a, a\} = [a^\dagger, a^\dagger\} = 0.$$

(2.5)

2.3. Algebra of generators

Let us now try to concoct a third symmetry generator G^3 from two known ones G^1 and G^2. If both commute with the S-matrix, then so does their product $G^1 G^2$. This product, however, is not a generator of a symmetry in the sense defined above since it is quadrilinear in particle operators. The canonical quantisation rules suggest that we try the commutator $[G^1, G^2]$.

We first do this for two bosonic generators B^1 and B^2. We can use the quantisation rules together with the identities

$$[ab, c] \equiv a[b, c] + [a, c]b \equiv a\{b, c\} - \{a, c\}b,$$

(2.6)

and after a bit of algebra we get

$$[B^1, B^2] = B^3,$$

(2.7a)

where the kernels of B^3, which define it, are given by

$$K_{bb}^3 = K_{bb}^1 * K_{bb}^2 - K_{bb}^2 * K_{bb}^1; \qquad K_{ff}^3 = K_{ff}^1 * K_{ff}^2 - K_{ff}^2 * K_{ff}^1. \tag{2.7b}$$

The quantisation rules have eliminated two of the four particle operators originally present in the product, and the commutator has turned out to be bilinear in particle operators. It is thus another generator of a symmetry.

Similarly, we can show that the commutator of an F with a B gives another F:

$$[F^1, B^2] = F^3 \tag{2.8a}$$

with

$$K_{fb}^3 = K_{fb}^1 * K_{bb}^2 - K_{ff}^2 * K_{fb}^1; \qquad K_{bf}^3 = K_{bf}^1 * K_{ff}^2 - K_{bb}^2 * K_{bf}^1. \tag{2.8b}$$

There is, however, no way to decompose the commutator $[b^\dagger f, f^\dagger b]$, which appears in $[F^1, F^2]$, in such a way that only those proper combinations $[..,..\}$ appear which allow elimination of two of the four particle operators in the product. We conclude that the commutator of two odd generators is not bilinear in particle operators and hence not a symmetry generator.

On the other hand, we find that we can evaluate the *anti*commutator of two F's, using the identities

$$\{ab, c\} \equiv a\{b, c\} - [a, c]b \equiv a[b, c] + \{a, c\}b \tag{2.9}$$

as well as the ones of eq. (2.6). The anticommutator of two F's turns out to be the generator of an even symmetry:

$$\{F^1, F^2\} = B^3 \tag{2.10a}$$

with

$$K_{bb}^3 = K_{bf}^1 * K_{fb}^2 + K_{bf}^2 * K_{fb}^1; \qquad K_{ff}^3 = K_{fb}^1 * K_{bf}^2 + K_{fb}^2 * K_{bf}^1. \tag{2.10b}$$

The B's and the F's satisfy the relations characteristic of a *graded Lie algebra*. It is the grading of the canonical quantisation rules, i.e., that the fermions obey anticommutation relations, which induces a similar grading for the symmetry generators. The odd generators behave like fermionic objects, the even ones like bosonic objects.

2.4. Graded Lie algebras

Let us explore some fundamental properties of graded Lie algebras. They are defined by the algebraic relations (2.7–10) which we now write in terms of a basis (B_i, F_α):

$$[B_i, B_j] = i c_{ij}{}^k B_k$$

$$[F_\alpha, B_i] = s_{\alpha i}{}^\beta F_\beta \tag{2.11}$$

$$\{F_\alpha, F_\beta\} = \gamma_{\alpha\beta}{}^i B_i.$$

The structure constants cannot be completely arbitrary. The c and γ have symmetry properties,

$$c_{ij}{}^k = -c_{ji}{}^k \; ; \qquad \gamma_{\alpha\beta}{}^i = \gamma_{\beta\alpha}{}^i , \tag{2.12}$$

and all structure constants are subject to consistency conditions which follows from the *graded Jacobi identities*

$$[[G^1, G^2\}, G^3\} + \text{graded cyclic} \equiv 0 . \tag{2.13}$$

The "graded cyclic sum" is defined just as the cyclic sum, except that there is an additional minus-sign if two fermionic operators are interchanged, e.g.:

$$F_\alpha F_\beta B_i + \text{graded cyclic} = F_\alpha F_\beta B_i + B_i F_\alpha F_\beta - F_\beta B_i F_\alpha . \tag{2.14a}$$

The full set of graded Jacobi identities is then

$$
\begin{aligned}
&[[B_i, B_j], B_k] + [[B_k, B_i], B_j] + [[B_j, B_k], B_i] = 0 \\
&[[F_\alpha, B_i], B_j] + [[B_j, F_\alpha], B_i] + [[B_i, B_j], F_\alpha] = 0 \\
&[\{F_\alpha, F_\beta\}, B_i] + \{[B_i, F_\alpha], F_\beta\} - \{[F_\beta, B_i], F_\alpha\} = 0 \\
&[\{F_\alpha, F_\beta\}, F_\gamma] + [\{F_\gamma, F_\alpha\}, F_\beta] + [\{F_\beta, F_\gamma\}, F_\alpha] = 0 .
\end{aligned}
\tag{2.14b}
$$

The requirement that these be fulfilled is equivalent to the demand that certain matrices constructed from the structure constants should form a representation r of the algebra, the *adjoint representation*. These matrices are

$$r(B_i) = \begin{bmatrix} C_i & 0 \\ 0 & S_i \end{bmatrix}, \qquad r(F_\alpha) = \begin{bmatrix} 0 & \Sigma_\alpha \\ \Gamma_\alpha & 0 \end{bmatrix} \tag{2.15a}$$

with matrix elements

$$
\begin{aligned}
&(C_i)_j{}^k = i c_{ji}{}^k \; ; \qquad (S_i)_\alpha{}^\beta = s_{\alpha i}{}^\beta \\
&(\Gamma_\alpha)_\beta{}^i = \gamma_{\beta\alpha}{}^i \; ; \qquad (\Sigma_\alpha)_i{}^\beta = s_{i\alpha}{}^\beta .
\end{aligned}
\tag{2.15b}
$$

The requirement that these matrices form a representation is equivalent to the following four conditions:

(a) the matrices C_i form a representation (the adjoint representation) of the bosonic *Lie subalgebra* which is spanned by the B's alone;

(b) the matrices S_i also form a representation of the Lie subalgebra (this representation need not be irreducible);

(c) the $\gamma_{\alpha\beta}{}^i$ are numerical invariants under the Lie subalgebra:

$$(S_i)_\alpha{}^\delta \gamma_{\delta\beta}{}^j + (S_i)_\beta{}^\delta \gamma_{\alpha\delta}{}^j - \gamma_{\alpha\beta}{}^k (C_i)_k{}^j = 0 ; \tag{2.16}$$

(d) there is a cyclic identity involving s and γ:

$$\gamma_{\alpha\beta}{}^i s_{\gamma i}{}^\delta + \text{cyclic} \, (\alpha \to \beta \to \gamma) = 0 . \tag{2.17}$$

2.5. The Coleman–Mandula theorem

Coleman and Mandula [8] studied the properties of all bosonic generators of symmetries in the mathematical framework of relativistic field theory. The assumptions they made (non-trivial S-matrix, etc.) can hardly be questioned from the point of view of physics, except perhaps for the "strong spectral assumption" that there should be only one zero-mass state, the unique vacuum, and a finite energy gap between it and the lowest one-particle state. This assumption, however, is introduced mainly to avoid infrared problems and relaxing it – in the hope that infrared problems will somehow take care of themselves – alters their result only in that the conformal group may then be admitted as symmetry group beyond the Poincaré group if all one-particle states are massless [34]. The assumption of a unique vacuum also appears to remove theories with spontaneously broken symmetries from their (and our) investigation. It could be interesting to explore the possibility of further symmetries which show discontinuous behaviour when the parameters of the model are changed: as long as the vacuum is unique they are absent, but they appear as soon as spontaneous breaking of some other symmetry introduces a degenerate vacuum.

The fact that only Lie groups of symmetries were considered excluded fermionic generators, and thus supersymmetry as we know it, from the analysis of Coleman and Mandula. Nevertheless, their results still hold for the subset of all bosonic generators and, through the conditions of the previous subsection, severely restrict the fermionic generators as well.

They found that any group of bosonic symmetries of the S-matrix in relativistic field theory is the *direct product* of the Poincaré group with an internal symmetry group. Furthermore, the latter must be the direct product of a compact semisimple group with U(1) factors.

The bosonic generators are thus the four momenta P_μ and the six Lorentz generators $M_{\mu\nu}$, plus a certain number of Hermitian internal symmetry generators B_r. The algebra is that of the Poincaré group,

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho) \tag{2.18}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho}) ,$$

together with that of the internal symmetry group,

$$[B_r, B_s] = i c_{rs}{}^t B_t . \tag{2.19}$$

The direct-product structure manifests itself in the vanishing of the commutators

$$[B_r, P_\mu] = [B_r, M_{\mu\nu}] = 0 , \tag{2.20}$$

in other words, the B_r must be translationally invariant Lorentz scalars.

The Casimir operators of the Poincaré group are the mass-square operator $P^2 \equiv P_\mu P^\mu$, and the generalised spin operator $W^2 \equiv W_\mu W^\mu$, where W^μ is the *Pauli–Lubanski* vector:

$$W^\mu = -\tfrac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} . \tag{2.21}$$

In the rest frame of a massive state we have

$$P_\mu = (m, 0, 0, 0) \quad \text{and} \quad W^2 = -m^2 L^2 \tag{2.22a}$$

with

$$L = (M_{23}, M_{31}, M_{12}). \tag{2.22b}$$

These Casimir operators commute with the entire Poincaré group and also with all internal symmetry generators:

$$[B_r, P^2] = 0; \quad [B_r, W^2] = 0. \tag{2.23}$$

The first of these equations says that all members of an irreducible multiplet of the internal symmetry group must have the same mass. This is also known as *O'Raifeartaigh's theorem* [49]. The second equation says that they must also have the same spin.

For massless states with discrete helicities we have

$$W_\mu = \lambda P_\mu \quad (\lambda \text{ half-integer}) \tag{2.24}$$

and $P^2 = W^2 = 0$. Again, no B_r can change the helicity since $[B_r, P_\mu] = [B_r, W_\mu] = 0$, and the statement that internal symmetries cannot change spin still holds. Stated in a positive way, the Coleman–Mandula no-go theorem reads:

All generators of supersymmetries must be fermionic, i.e., they must change the spin by a half-odd amount and change the statistics of the state.

Witten [77] has given a beautiful explanation for the essential content of the Coleman–Mandula theorem: additional space–time symmetries beyond energy, momentum and angular momentum would, in a relativistically covariant theory, overconstrain the elastic scattering amplitudes and allow non-zero scattering amplitudes only for discrete scattering angles. The assumption of *analyticity* therefore rules out such symmetries.

2.6. The supersymmetry algebra

This leaves us to explore the possibilities of fermionic generators, the odd elements of the graded algebra.

The Jacobi identities (subsection 2.4) link the properties of the fermionic sector of the algebra closely to those of the bosonic sector. The Coleman–Mandula theorem therefore provides strong limitations for the fermionic generators as well. The results of detailed examination [34] of these limitations are often referred to as the "Haag–Łopuszański–Sohnius theorem".

In ref. [34] a "positive metric assumption" was made about the Hilbert space. This means that the anticommutator of an operator with its adjoint is non-negative, and zero only if the operator is zero itself:

$$\langle ..|\{Q, Q^\dagger\}|.. \rangle = |Q^\dagger|.. \rangle|^2 + |Q|.. \rangle|^2 > 0 \quad \text{if } Q \neq 0. \tag{2.25}$$

If there is no element of the bosonic subset of the generators that possesses all the properties required of $\{Q, Q^\dagger\}$, we conclude that $Q = 0$. This line of argument is at the nucleus of most of the statements in ref. [34].

We already know from the general considerations of subsection 2.4 that the supersymmetry generators, which from now on are called Q (rather than F) in order to follow general usage, must carry a representation of the bosonic symmetry group. If Q sits in the representation (j, j') of the Lorentz group, then $\{Q, Q^\dagger\}$ will contain the representation $(j + j', j + j')$. Since P_μ is the only object in the bosonic sector which is in such a representation, namely in $(\frac{1}{2}, \frac{1}{2})$, we know that all Q's must be in one of the two 2-dimensional representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the algebra of the Lorentz group. Thus we have

$$[Q_{\alpha i}, M_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta i}$$

$$[\bar{Q}^i{}_{\dot\alpha}, M_{\mu\nu}] = -\tfrac{1}{2}\bar{Q}^i{}_{\dot\beta} (\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}$$

(2.26)

with σ-matrices as defined in the Technical Appendix, eqs. (A.14–17).

The index i in $Q_{\alpha i}$ labels all the different 2-spinors Q_α that are present and runs from 1 to some integer N. We assume that if Q is the generator of a symmetry then so is $Q^\dagger$. Since $(Q_{\alpha i})^\dagger$ must be in the complex conjugate representation, and thus be one of the $\bar Q$'s, we number those in such a way that always

$$\bar{Q}^i{}_{\dot\alpha} = (Q_{\alpha i})^\dagger .$$

(2.27)

The anticommutator $\{Q, \bar{Q}\}$ transforms as $(\frac{1}{2}, \frac{1}{2})$ and must therefore be proportional to the energy-momentum operator:

$$\{Q_{\alpha i}, \bar{Q}^j{}_{\dot\beta}\} = 2\delta_i{}^j (\sigma^\mu)_{\alpha\dot\beta} P_\mu .$$

(2.28)

The factor $\delta_i{}^j$ is a matter of properly defining the Q's. We would at first assume that

$$\{Q_{\alpha i}, \bar{Q}^j{}_{\dot\beta}\} = 2c_i{}^j (\sigma^\mu)_{\alpha\dot\beta} P_\mu$$

with some coefficients $c_i{}^j$. Since the matrices σ^μ are Hermitian (see below), we find, by taking the adjoint, that $c_i{}^j = (c_j{}^i)^*$, i.e., that the matrix C with matrix elements $c_i{}^j$ is Hermitian. Then there is a unitary matrix U which diagonalises C, and we redefine

$$Q_{\alpha i} \to U_i{}^j Q_{\alpha i} ; \qquad \bar{Q}^i{}_{\dot\alpha} \to \bar{Q}^j{}_{\dot\alpha} (U^{-1})_j{}^i .$$

If the energy is to be positive (see below), all eigenvalues c_i of the matrix C must be positive and we can rescale

$$Q_{\alpha i} \to \sqrt{c_i}\, Q_{\alpha i} ; \qquad \bar{Q}^i{}_{\dot\alpha} \to \sqrt{c_i}\, \bar{Q}^i{}_{\dot\alpha} \qquad \text{(no sum)}$$

and get (2.28) for the twice redefined Q's. The redefinitions have not spoiled the reality connection (2.27) between Q and $\bar{Q}$.

Knowing that the structure constants on the right-hand side of eq. (2.28) must be numerically

invariant tensors of the Lorentz group, see eq. (2.16), we can conclude that the matrices σ^μ must be the 2×2 matrices defined in the Technical Appendix, eq. (A.28), or multiples thereof. We have normalised the Q's so that the absolute value of the factor in (2.28) is 2. Another widely used and perhaps more natural normalisation would be 1. The *sign* on the right-hand side of (2.28) is determined by the requirement that the energy $E \equiv P_0$ should be a positive definite operator: we get for each value of the index i

$$\sum_{\alpha=1}^{2} \{Q_{\alpha i}, (Q_{\alpha i})^\dagger\} = 2 \operatorname{tr} \sigma^\mu P_\mu = 4P_0 \qquad \text{(no sum over } i), \tag{2.29}$$

and we find from (2.25) that the energy is indeed positive definite for our choice of sign (this depends on the metric; authors who use $\eta_{00} = -1$ may have a minus sign, depending on their other conventions).

The supersymmetry generators commute with the momenta,

$$[Q_{\alpha i}, P_\mu] = [\bar{Q}^i{}_{\dot\alpha}, P_\mu] = 0. \tag{2.30}$$

Because of its importance, the proof for this equation follows:

The commutator of a Q with a P_μ could contain the Lorentz representations $(1, \frac{1}{2})$ and $(0, \frac{1}{2})$. Since there are no $(1, \frac{1}{2})$ generators present, we get as the most general possibility a sum over $\bar{Q}$'s (see the Technical Appendix for some of the conventions used):

$$[Q_{\alpha i}, P_\mu] = c_{ij}(\sigma_\mu)_{\alpha\dot\beta}\bar{Q}^{\dot\beta j}.$$

The adjoint of this is

$$[\bar{Q}^{\dot\alpha i}, P_\mu] = (c_{ij})^* (\bar\sigma_\mu)^{\dot\alpha\beta} Q_{\beta j},$$

so that

$$[[Q_{\alpha i}, P_\mu], P_\nu] = c_{ij} (c_{jk})^* (\sigma_\mu \bar\sigma_\nu)_\alpha{}^\beta Q_{\beta k}.$$

From $[P_\mu, P_\nu] = 0$ and the Jacobi identity we deduce that for the matrix C with matrix elements c_{ij}:

$$CC^* = 0,$$

since the σ-matrix factor does not vanish. The most general $\{Q, Q\}$ that we can have is

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\varepsilon_{\alpha\beta}Z_{[ij]} + \text{a term symmetric in } \alpha\beta.$$

The Z_{ij} are a linear combination of internal symmetry generators and hence commute with P_μ. Then

$$0 = \varepsilon^{\alpha\beta}[\{Q_{\alpha i}, Q_{\beta j}\}, P_\mu] = \varepsilon^{\alpha\beta}\{Q_{\alpha i}, [Q_{\beta j}, P_\mu]\} - (i \leftrightarrow j)$$

$$= \varepsilon^{\alpha\beta}c_{jk}(\sigma_\mu)_{\beta\dot\beta}\{Q_{\alpha i}, \bar{Q}^{\dot\beta k}\} - (i \leftrightarrow j) \propto c_{[ij]}P_\mu$$

which means that C is symmetric, hence $CC^\dagger = 0$ which implies $c_{ij} = 0$, q.e.d.

The Q's generally also carry some representation of the internal symmetry,

$$[Q_{\alpha i}, B_r] = (b_r)_i{}^j Q_{\alpha j}.$$ (2.31a)

Since the internal symmetry group is compact, the representation matrices can be chosen Hermitian, $b_r = b_r^\dagger$. For $\bar{Q}$ we then have

$$[\bar{Q}^i{}_{\dot\alpha}, B_r] = -\bar{Q}^j{}_{\dot\alpha}(b_r)_j{}^i.$$ (2.31b)

The largest possible internal symmetry group which can act non-trivially on Q is thus $U(N)$.

The double index structure on the Q's reflects the direct product structure of the bosonic symmetry group: α and $\dot\alpha$ belong to the Poincaré group and i is the internal symmetry index.

The last algebraic relation which we have to consider is the anticommutator of Q with Q (that of $\bar{Q}$ with $\bar{Q}$ can then be obtained by Hermitian conjugation). Lorentz covariance requires $\{Q, Q\}$ to be a linear combination of bosonic operators in the representations $(0, 0)$ and $(1, 0)$ of the Lorentz group. The only $(1, 0)$ present in the bosonic sector of the algebra is the self-dual part of $M_{\mu\nu}$. Such a term in $\{Q, Q\}$ would, however, not commute with P_μ whereas $\{Q, Q\}$ does, due to eq. (2.30). Thus we are left with the following most general form for the anticommutator:

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\varepsilon_{\alpha\beta}Z_{ij},$$ (2.32)

where the Z_{ij} are some linear combination of the internal symmetry generators:

$$Z_{ij} = a^r_{ij}B_r.$$ (2.33)

We now proceed to show that

$$[Z_{ij}, \text{anything}] = 0,$$ (2.34)

which has given rise to the name *central charges* for the Z's.

From eqs. (2.31) and (2.32) we get

$$[Z_{ij}, B_r] = (b_r)_i{}^k Z_{kj} + (b_r)_j{}^k Z_{ik}$$

and then, with (2.33),

$$[Z_{ij}, Z_{kl}] = a_{kl}{}^r (b_r)_i{}^k Z_{kj} + a_{kl}{}^r (b_r)_j{}^k Z_{ik}.$$

These two equations say that the Z_{ij} span an invariant subalgebra of the internal symmetry algebra. With the help of the Jacobi identity, the commutator $[\bar{Q}, Z_{ij}]$ can be rewritten as a sum of $[Q, P_\mu]$ commutators and thus vanishes. This implies

$$a_{ij}{}^r (b_r)_k{}^l = 0$$

and therefore $[Q, Z_{ij}] = [Z_{ij}, Z_{kl}] = 0$, and the invariant subalgebra is Abelian. The direct product

structure (semisimple $\otimes$ Abelian) of the internal symmetry group now implies that all Z_{ij} lie in the Abelian factor. Thus

$$[B_r, Z_{ij}] = 0,$$

and Z_{ij} commutes with everything.

The adjoint of (2.32) is (see eq. (A.19) for the extra minus sign):

$$\{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = -2\varepsilon_{\dot\alpha\dot\beta} Z^{ij} \qquad \text{with} \qquad Z^{ij} = (Z_{ij})^\dagger. \tag{2.35}$$

The Z^{ij} are, of course, also central charges. The symmetry of $\{Q, Q\}$ under interchange of αi with βj and the antisymmetry of $\varepsilon_{\alpha\beta}$ mean that

$$Z_{ij} = -Z_{ji} \qquad \text{i.e.} \qquad a^r_{ij} = -a^r_{ji}, \tag{2.36}$$

which excludes central charges for $N = 1$. Furthermore, for each non-vanishing central charge there must be a different antisymmetric $N \times N$ matrix a^r which is a numerical invariant of the internal symmetry group:

$$(b^s)_i{}^k a^r_{kj} + (b^s)_j{}^k a^r_{ik} = 0 \tag{2.37}$$

(this follows from eq. (2.16)). Central charges in the algebra therefore impose a *symplectic structure* on the semi-simple part of the internal symmetry group. The largest internal symmetry group which can sustain a central charge is USp(N), the compact version of the symplectic group Sp(N).

2.7. Summary of algebra

The following equations summarise the results of the previous subsection:

$$[P_\mu, P_\nu] = 0 \tag{2.38a}$$

$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho) \tag{2.38b}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}) \tag{2.38c}$$

$$[B_r, B_s] = ic_{rs}{}^t B_t \tag{2.38d}$$

$$[B_r, P_\mu] = [B_r, M_{\mu\nu}] = 0 \tag{2.38e}$$

$$[Q_{\alpha i}, P_\mu] = [\bar{Q}^i_{\dot\alpha}, P_\mu] = 0 \tag{2.38f}$$

$$[Q_{\alpha i}, M_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta i} \tag{2.38g}$$

$$[\bar{Q}^i_{\dot\alpha}, M_{\mu\nu}] = -\tfrac{1}{2}\bar{Q}^i_{\dot\beta}(\bar{\sigma}_{\mu\nu})^{\dot\beta}{}_{\dot\alpha} \tag{2.38h}$$

$$[Q_{\alpha i}, B_r] = (b_r)_i{}^j Q_{\alpha j}$$
(2.38i)

$$[\bar{Q}^i{}_{\dot{\alpha}}, B_r] = -\bar{Q}^j{}_{\dot{\alpha}} (b_r)_j{}^i$$
(2.38j)

$$\{Q_{\alpha i}, \bar{Q}^j{}_{\dot{\beta}}\} = 2\delta^j_i (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$$
(2.38k)

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\varepsilon_{\alpha\beta} Z_{ij} \quad \text{with} \quad Z_{ij} = a^r_{ij} B_r$$
(2.38l)

$$\{\bar{Q}^i{}_{\dot{\alpha}}, \bar{Q}^j{}_{\dot{\beta}}\} = -2\varepsilon_{\dot{\alpha}\dot{\beta}} Z^{ij} \quad \text{with} \quad Z^{ij} = (Z_{ij})^\dagger$$
(2.38m)

$$[Z_{ij}, \text{anything}] = 0 .$$
(2.38n)

This concludes the discussion of possible supersymmetry algebras which have the Poincaré group as space–time symmetry. As already mentioned, it is possible to include the conformal group into the bosonic sector if all masses are zero. This results in what is known as *conformal supersymmetry*, a subject covered in section 16.

If there is only one 2-spinor supercharge Q_α, i.e., if $N = 1$, we say that a theory exhibits *simple* or *unextended supersymmetry*. If $N > 1$, we speak of *extended supersymmetry*. For simple supersymmetry the only non-trivially acting internal symmetry is a single U(1), generated by a charge which has become known under the name of R:

$$[Q, R] = Q; \quad [\bar{Q}, R] = -\bar{Q} .$$
(2.39)

Since under parity $Q \to \bar{Q}$ and $\bar{Q} \to Q$, we must have $R \to -R$, i.e., the U(1) symmetry group is *chiral* (this will be much clearer in 4-spinor notation).

2.8. Cautionary remarks

In the course of the discussion so far, it has become increasingly clear that supersymmetries, which were originally defined rather loosely as any symmetries that involve different spins, can appear in relativistic field theories only in a very specific form: they are generated by spinorial charges Q which observe well-defined anticommutation relations. It is this scheme which is generally meant when we speak of "supersymmetry" or – in the context of curved space–time – of "supergravity".

The restrictions of the Coleman–Mandula theorem have turned out to be considerably less stringent for fermionic symmetry generators: in particular, we now have

$$[Q, W^2] \neq 0 .$$

The *O'Raifeartaigh theorem* [49], however,

$$[Q, P^2] = 0 ,$$

still holds even for the spinorial charges, since $[Q, P_\mu] = 0$. Supersymmetry multiplets therefore contain different spins but are always degenerate in mass and supersymmetry must be broken in nature where elementary particles do not come in mass-degenerate multiplets.

I must point out another limitation of the discussion in both refs. [8] and [34]: these papers deal with *symmetries of the S-matrix*, i.e., of the observed scattering data. The underlying field theory may have more, or less, symmetry.

A large part of the analysis in [34] is based on the positivity assumption of the metric in Hilbert space, as in eq. (2.22). This and the spin-statistics theorem, which is used throughout, rule out statements about, say, the ghost sector of a quantised gauge theory and its BRS transformations.

Obviously, since refs. [8] and [34] deal with Poincaré invariant theories in flat Minkowski space, they do not strictly speaking apply to gravity or supergravity. Hence we are not surprised to learn that, for instance, non-compact internal symmetries appear in some extended supergravity theories.

We see that whatever no-go statements are contained in the present section should be taken with a grain of salt. Still, we shall now proceed to discuss theories which are supersymmetric in the limited sense outlined so far. In fact, very little is known about possible infringements on the no-go theorems which limit supersymmetry algebras.

3. Representations of the supersymmetry algebra

Supersymmetric theories deal with sets of fields or states which carry representations of one of the described graded Lie algebras (supersymmetry algebras). Loosely speaking, a representation consists of linear operators on some vector space which have the same "properties" as the objects which they are to represent. In our case, their commutator–anticommutator relations should be those of the supersymmetry algebra.

3.1. The "fermions = bosons" rule

The very nature of the graded algebras demands a grading of the vector space on which a representation acts. In less fancy language: since supersymmetry transformations relate fermions and bosons, the representation space can be divided into a bosonic and a fermionic sector:

$$\left(\begin{array}{c} \text{representation} \\ \text{space} \end{array} \right) = \left(\begin{array}{c} \text{bosonic} \\ \text{subspace} \end{array} \right) + \left(\begin{array}{c} \text{fermionic} \\ \text{subspace} \end{array} \right) \tag{3.1}$$

The even elements of the supersymmetry algebra (the bosonic generators $P_\mu, M_{\mu\nu}, B_r$) map the subspaces into themselves, and a-priori we must allow the mapping to be onto a proper subspace of the original subspace,

bosonic/fermionic   bosonic/fermionic
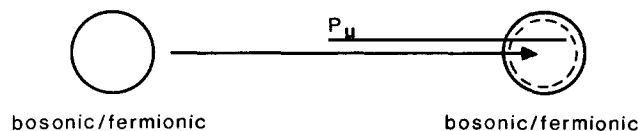
The odd elements (the supersymmetry generators) map the bosonic subspace into the fermonic one and vice-versa, i.e.:

bosonic/fermionic   fermionic/bosonic
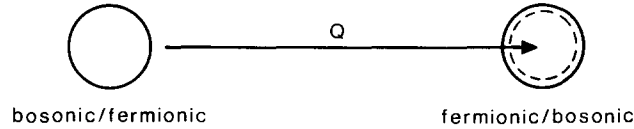
The anticommutators are a particular linear combination of two subsequent mappings which can be symbolised like this:

bosonic/fermionic   fermionic/bosonic   bosonic/fermionic
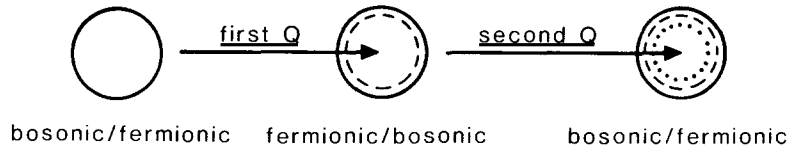
The relation $\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu$, characteristic of supersymmetry, means that there should be a particular way of going from one subspace to the other and back, such that the net result is as if we had operated with P_μ on the original subspace.

For a large class of representations, the representation $r(P_\mu)$ of the translation operator will map the representation space onto itself. On a physical multiparticle state, for instance, we have

$$r(P_\mu) = (E, p),\tag{3.2}$$

with E and p the total energy and momentum of the state. On quantum fields $\phi(x)$, P_μ is the generator of translations $\phi(x) \rightarrow \phi(x + y)$, and is represented by the partial derivatives

$$r(P_\mu) = i\partial_\mu.\tag{3.3}$$

In both cases, no "degrees of freedom" are lost from the representation space through the action of P_μ. This implies that the dimension of the fermionic subspace must be the same as that of the bosonic subspace. This is so because

(a) no picture can have more dimensions than its original, and

(b) in the double mapping $Q\bar{Q} + \bar{Q}Q$ no dimensions get lost.

We thus have the following theorem which holds for the wide class of representations characterised by (3.2) or (3.3):

The fermionic and bosonic subspaces of the representation space of a supersymmetry algebra have the same dimension.

What exactly makes up the "dimension" of such a representation space remains to be seen.

Historically, this property of representations quickly became an established rule of thumb. Since there is, of course, no need to ever postulate the property – it simply follows from the fact that a given multiplet carries a representation – no complete proof for all possible cases has to my knowledge been attempted. Hopefully, the somewhat loose argumentation above will make the reader appreciate this

"fermions = bosons" rule which has played such an important role in the search for what is known as "off-shell representations".

A couple of cases where the rule does not hold ought to be mentioned. The *adjoint representation* of the superalgebra, see subsection 2.4, has $r(P_\mu) = 0$, and thus is not covered by the rule. Its representation space are the generators themselves, and usually there are indeed a different number of fermionic and bosonic generators. Another exception are *non-linear realisations*, where nothing of the above arguments applies: we don't have a representation vector space and cannot speak of "dimension". In fact, there are several ways to realise the $N = 1$ supersymmetry algebra non-linearly on a single chiral spinor and its adjoint [70, 78].

3.2. Massless one-particle states

We now turn our attention to representations of the supersymmetry algebra on one-particle states [56]. Since the most interesting supersymmetric models are extensions of non-Abelian gauge theories and of quantised gravity, all of which have zero-mass exchange particles and hence whole massless supermultiplets, the representations on massless states are particularly important. They allow to understand the "physical" spectra of the models.

A massless one-particle state can always be rotated into a *standard frame* where its movement is in the z-direction, i.e. where

$$P_\mu = (E, 0, 0, E).\tag{3.4}$$

The space–time properties of the state are then determined by its *energy* E and its *helicity* λ. The helicity is the projection of its spin onto the direction of motion, i.e. the eigenvalue of $E^{-1}L \cdot P$. From the definitions of the Pauli–Lubanski vector and of the angular momentum, eqs. (2.21) and (2.22b), we find that $W_0 = L \cdot P$. For a massless helicity eigenstate $|E, \lambda\rangle$ we then always have $W_0 = \lambda E$, or, because of Lorentz covariance,

$$W_\mu |E, \lambda\rangle = \lambda P_\mu |E, \lambda\rangle.\tag{3.5}$$

If we let one of the operators $Q_{\alpha i}$ act on the states, energy and momentum remain unchanged since $[Q, P_\mu] = 0$. The helicity of the resulting state can be determined as follows:

$$W_0 Q_\alpha |E, \lambda\rangle = Q_\alpha W_0 |E, \lambda\rangle + [W_0, Q_\alpha]|E, \lambda\rangle = E(\lambda \mathbb{1} - \tfrac{1}{2}\sigma^3)_\alpha{}^\beta Q_\beta |E, \lambda\rangle.$$

We have used the particular form (3.4) of the momentum and $[L \cdot P, Q] = -\tfrac{1}{2}\sigma \cdot PQ$, which follows from eqs. (2.22b) and (2.38g), together with our particular representation of $\sigma_{\mu\nu}$. Substituting the explicit form of σ^3, we see that all Q_{1i} lower the helicity by $\tfrac{1}{2}$ and all Q_{2i} raise it. For $\bar{Q}_{\dot{\alpha}}$ the calculation is similar, and due to the minus sign in eq. (2.38h) the result is the opposite: all $\bar{Q}_{\dot{1}}$ raise the helicity by $\tfrac{1}{2}$ and all $\bar{Q}_{\dot{2}}$ lower it.

Using the explicit form (A.28) for the matrices σ^μ, we find that in the absence of central charges the algebra of the Q's on our standard frame states reduces to

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0; \qquad \{Q_{2i}, \bar{Q}^j{}_{\dot{2}}\} = 0$$
$$\{Q_{1i}, \bar{Q}^j{}_{\dot{1}}\} = 4\delta_i{}^j E,\tag{3.6a}$$

or

$$\{q_i, \bar{q}^j\} = \delta_i{}^j; \qquad \{q_i, q_j\} = \{\bar{q}^i, \bar{q}^j\} = 0 \tag{3.6b}$$

if we rescale

$$q_i \equiv (4E)^{-1/2} Q_{1i}. \tag{3.6c}$$

From our positivity assumption, as expressed by eq. (2.25), we conclude that

$$Q_{2i} = 0 \tag{3.7}$$

on these states. The remaining algebra (3.6b) of the q_i is the Clifford algebra for N fermionic degrees of freedom. Any *irreducible* representation of this algebra is characterised by a Clifford *ground state* $|E, \lambda_0\rangle$ with

$$q_i|E, \lambda_0\rangle = 0, \tag{3.8}$$

and the other states are generated by successive application of the N operators $\bar{q}^i$:

$$\bar{q}^i|E, \lambda_0\rangle = |E, \lambda_0 + \tfrac{1}{2}, i\rangle$$

$$\bar{q}^i\bar{q}^j|E, \lambda_0\rangle = |E, \lambda_0 + 1, ij\rangle, \tag{3.9}$$

etc., until we have reached the "top" state

$$\bar{q}^1\bar{q}^2\cdots\bar{q}^N|E, \lambda_0\rangle = |E, \lambda_0 + n/2, 1\ 2\cdots N\rangle \tag{3.10}$$

on which any further application of a $\bar{q}$ will give zero. In addition to whatever other labels (internal symmetry eigenvalues) the ground state may have, the other states also carry the internal symmetry labels of the $\bar{q}$'s which generated them. They must be totally antisymmetric in these labels: $|E, \lambda, ij\rangle = -|E, \lambda, ji\rangle$, etc. The helicity, of course, has been raised by $\tfrac{1}{2}$ in each application of a $\bar{q}^i$. We thus have the following spectrum:

helicity: λ_0 $\lambda_0 + \tfrac{1}{2}$ $\lambda_0 + 1$ $\cdots$ $\lambda_0 + n/2$

number of states: $\binom{N}{0} = 1$ $\binom{N}{1} = N$ $\binom{N}{2}$ $\cdots$ $\binom{N}{N} = 1$. $\tag{3.11}$

The total range of helicities is from λ_0 to $\lambda_0 + N/2$. The total number of states and the "fermions = bosons" rule follow from properties of the binomial coefficients:

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N; \qquad \sum_{k=0}^{N/2} \binom{N}{2k} - \sum_{k=0}^{N/2} \binom{N}{2k+1} = 0.$$

The first of these equations is the binomial expansion of $(1 + 1)^N$ and the second is that of $(1 - 1)^N$.

Two particularly important examples of these spectra are the $N = 4$ Yang–Mills multiplet with $\lambda_0 = -1$ and the $N = 8$ supergravity multiplet with $\lambda_0 = -2$:

$$\underline{N = 4} \quad \text{helicity:} \quad -1 \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \quad 1$$

$$\text{states:} \quad 1 \quad 4 \quad 6 \quad 4 \quad 1, \tag{3.12}$$

$$\underline{N = 8} \quad \text{helicity:} \quad -2 \quad -\tfrac{3}{2} \quad -1 \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \quad 1 \quad \tfrac{3}{2} \quad 2$$

$$\text{states:} \quad 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1. \tag{3.13}$$

Normally, any spectrum of states that is derived from a Lorentz-covariant field theory will exhibit *PCT*-symmetry [42]. This implies that for every state with helicity λ there should be a parity reflected state which has helicity $-\lambda$. Our spectra do not generally have this property, e.g. for $N = 1$, $\lambda_0 = 0$, we get

$$\text{helicity:} \quad 0 \quad \tfrac{1}{2}$$

$$\text{states:} \quad 1 \quad 1.$$

The spectrum of a Lorentz-covariant field theory can therefore contain these states only in conjunction with those of the *PCT*-conjugate multiplet with $\lambda_0 = -\tfrac{1}{2}$,

$$\text{helicity:} \quad -\tfrac{1}{2} \quad 0$$

$$\text{states:} \quad 1 \quad 1,$$

so that the smallest $N = 1$ multiplet, even for the massless case, is:

$$\underline{N = 1} \quad \text{helicity:} \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2}$$

$$\text{states:} \quad 1 \quad 2 \quad 1. \tag{3.14}$$

We shall see later that this is the spectrum of the *massless Wess–Zumino model* [72], a model field theory which contains a scalar, a pseudoscalar and a Majorana spin-$\tfrac{1}{2}$ field in interaction.

It is a particular property of the $N = 4$ Yang–Mills theory and of $N = 8$ supergravity that their multiplets are actually *PCT-self-conjugate*.

Spin $\tfrac{3}{2}$ does not allow renormalisable coupling and spin $\tfrac{5}{2}$ does not allow consistent coupling to gravity. Therefore we see already at this stage that there are stringent limits on the number N of independent supersymmetries:

$N \leq 4$ for *renormalisable* theories;

$N \leq 8$ for consistent theories of *supergravity*.

If these limits were violated, the spectrum of massless states would contain helicities $\tfrac{3}{2}$ and $\tfrac{5}{2}$, respectively.

3.3. Massive one-particle states without central charges

The space–time properties of massive one-particle states are described by the mass m, the total spin s and the spin projection s_3 along the z-axis. We assume the particles to be in the *rest frame* where

$$P_\mu = (m, 0, 0, 0) .$$
(3.15)

Q is a *tensor operator of spin* $\frac{1}{2}$, as can be seen from $[Q, L] = \frac{1}{2}\sigma Q$. Therefore the result of the action of Q on a state with spin s will be a linear combination of states with spins $s + \frac{1}{2}$ and $s - \frac{1}{2}$:

$$Q|m\,s\,s_3\rangle = \sum_{s_3'} c^{(+)}_{s_3 s_3'}|m\ s+\tfrac{1}{2}\ s_3'\rangle + \sum_{s_3'} c^{(-)}_{s_3 s_3'}|m\ s-\tfrac{1}{2}\ s_3'\rangle .$$

The same is true for $\bar{Q}$ with, of course, different coefficients.

In the rest frame, the algebra without central charges reduces to

$$\{Q_{\alpha i}, \bar{Q}^j{}_{\dot\beta}\} = 2m\delta_i{}^j\,\delta_{\alpha\beta} ; \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0$$
(3.16a)

or

$$\{q_{\alpha i}, \bar{q}^j{}_{\dot\beta}\} = \delta_{\alpha\beta}\delta_i{}^j ; \quad \{q_i, q_j\} = \{\bar{q}^i, \bar{q}^j\} = 0$$
(3.16b)

if we rescale

$$q_{\alpha i} \equiv (2m)^{-1/2} Q_{\alpha i} .$$
(3.16c)

This is the Clifford algebra for $2N$ fermionic degrees of freedom. A representation is therefore characterised by a spin multiplet of *ground states* $|ms_0 s_3\rangle$, $s_3 = -s_0, \ldots +s_0$, which are annihilated by $q_{\alpha i}$. The other states are generated by successive application of $\bar{q}$'s to the ground states. A typical state would be

$$| \ \rangle = \bar{q}^{i_1}_{\dot\alpha_1} \cdots \bar{q}^{i_n}_{\dot\alpha_n} |ms_0 s_3\rangle .$$
(3.17)

The states are totally antisymmetric under interchange of index-pairs $(\dot\alpha i) \leftrightarrow (\dot\beta j)$. The *maximal spin* is carried by states like

$$\bar{q}^1_{\dot 1} \bar{q}^2_{\dot 1} \cdots \bar{q}^N_{\dot 1} |ms_0\rangle$$

and is

$$s_{\text{max}} = s_0 + N/2 .$$
(3.18)

The *minimal spin* is 0 if $N/2 \ge s_0$ or $s_0 - N/2$ otherwise. The "top" state is reached after all $2N$ operators $\bar{q}$ have been applied once. It always has spin s_0, since it can be seen as produced by application of the operator pairs $\bar{q}_{\dot 1}^1 \bar{q}_{\dot 2}^1, \bar{q}_{\dot 1}^2 \bar{q}_{\dot 2}^2$, etc., which all carry spin 0.

The dimension of a representation with $s_0 = 0$ is 2^{2N} because there are now $2N$ independent raising operators, compared with only N in the massless case.

Since renormalisability requires massive matter to have spin $\leq \frac{1}{2}$, we deduce from the above expression for s_{max} that we must have

$N = 1$ for *renormalisable* coupling of *massive matter.*

We see that, apart from supergravity where renormalisability is not an issue and in the absence of central charges, the only important multiplet with mass is that of the *massive Wess–Zumino model* [72], which has $N = 1$ and $s_0 = 0$ and contains a scalar, a pseudoscalar and the two spin states of a massive Majorana spinor:

$$\underline{N = 1} \qquad s^P: \qquad 0^+ \quad \tfrac{1}{2} \quad 0^-$$

$$\text{states:} \quad 1 \quad 2 \quad 1 . \tag{3.19}$$

Since, as we shall see, representations of the supersymmetry algebra on fields are formally very similar to those on massive states, the study of the latter is much more worthwhile than it would appear at this stage.

3.4. One-particle representations with central charges

The central charges Z_{ij} on the right-hand side of the $\{Q, Q\}$ anticommutator (2.38l) commute with everything, see eq. (2.38n). We can therefore choose a basis in our representation space where they are diagonal and represented by complex numbers z_{ij}. These form an antisymmetric $N \times N$ matrix which can be brought into a standard form with the help of a unitary matrix U:

$$\bar{z}_{ij} = U_i{}^k U_j{}^l z_{kl} . \tag{3.20}$$

The standard form is, for even N,

$$\bar{z} = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} \tag{3.21}$$

where D is a real, diagonal matrix with non-negative eigenvalues

$$0 \leq z_{(r)}; \qquad r = 1, \ldots N/2 . \tag{3.22}$$

If N is odd, there is an additional row and column in (3.21) with all zeros. We now use the unitary matrix U to redefine our Q's,

$$U_i{}^j Q_{\alpha j} \rightarrow Q_{\alpha i}; \qquad \bar{Q}^j{}_{\dot\alpha} (U^{-1})_j{}^i \rightarrow \bar{Q}^i{}_{\dot\alpha} ,$$

and introduce double-indices $i = (a, r)$ compatible with the obvious form of (3.21), i.e. $a = 1, 2$ and $r = 1, \ldots, N/2$. Again, for odd N, the last charge $Q_{\alpha N}$ is not touched by this.

The algebra of the Q's is now

$$\{Q_{\alpha ar}, \bar{Q}^{bs}_{\dot{\beta}}\} = 2\delta_a{}^b \, \delta_r{}^s \, (\sigma^\mu)_{\alpha\dot{\beta}} \, P_\mu \qquad\qquad (3.23a)$$

$$\{Q_{\alpha ar}, Q_{\beta bs}\} = 2\varepsilon_{\alpha\beta}\varepsilon_{ab}\delta_{rs}z_{(r)} \qquad\qquad (3.23b)$$

$$\{\bar{Q}^{ar}_{\dot{\alpha}}, \bar{Q}^{bs}_{\dot{\beta}}\} = -2\varepsilon_{\dot{\alpha}\dot{\beta}} \, \varepsilon^{ab}\delta^{rs} z_{(r)} \, . \qquad\qquad (3.23c)$$

For odd N, we also have

$$\{Q_{\alpha N}, Q_{\beta i}\} = 0; \qquad \{Q_{\alpha N}, \bar{Q}^i{}_{\dot{\beta}}\} = 2\delta_N{}^i \, (\sigma^\mu)_{\alpha\dot{\beta}}P_\mu \, . \qquad\qquad (3.23d)$$

Let us first consider the massless case. As before, we find in the standard frame (3.4) that $Q_{2i} = 0$. But now this implies, through (3.23b), that all $z_{(r)} = 0$ and we conclude that

massless particle representations represent central charges trivially.

For the massive case, we introduce linear combinations

$$A^{\pm}_{\alpha r} \equiv \tfrac{1}{2}(Q_{\alpha 1 r} \pm \bar{Q}^{\dot{\alpha}2r}) \qquad\qquad (3.24)$$

and their Hermitian adjoints. We notice how dotted and undotted Lorentz indices are mixed in such a way that covariance under the rotation subgroup is maintained since Q_α and $\bar{Q}^{\dot{\alpha}}$ transform in the same way under it, see eqs. (A.15) and (A.17).

In terms of $A^{\pm}$, the rest-frame algebra (3.23) now reads

$$\{A^{\pm}, A^{\pm}\} = \{A^{\pm}, A^{\mp}\} = \{A^{\pm}, (A^{\mp})^{\dagger}\} = 0$$

$$\{A^{\pm}_{\alpha r}, (A^{\pm}_{\beta s})^{\dagger}\} = \delta_{\alpha\beta}\delta_{rs}(m \pm z_{(r)}) \qquad\qquad (3.25)$$

and we conclude immediately, from the positivity of the left-hand side of the last equation, that

$$z_{(r)} \leq m \, . \qquad\qquad (3.26)$$

Let us assume that this bound is satisfied for a number n_0 of eigenvalues $z_{(r)}$ of the central charges. Then the corresponding A^- are represented trivially, and after rescaling the remaining generators

$$q^{\pm}_{\alpha r} \equiv (m \pm z_{(r)})^{-1/2}A^{\pm}_{\alpha r}$$

$$q_{\alpha N} \equiv m^{-1/2}Q_{\alpha N} \qquad \text{(if } N \text{ odd)} \, , \qquad\qquad (3.27)$$

we have the Clifford algebra for $2(N - n_0)$ fermionic degrees of freedom. As far as the spectrum is concerned, we have the same situation as without central charges, except that

N is effectively reduced by n_0, the number of central charges that satisfy the bound $m = z$.

The simplest representation with central charge, the $N = 2$ *hypermultiplet* [57, 14, 60] has one central charge which saturates the bound, and the spectrum is a doubled version of the massive Wess–Zumino model.

3.5. Spectrum generating groups [21]

We saw in subsection 3.2 that the massless states which are generated by applying n operators $\bar{q}^i \propto \bar{Q}^i_{\dot{1}}$ to the ground state $|E, \lambda_0\rangle$ carry n antisymmetric indices $i_1, \ldots i_n$. There are $\binom{N}{n}$ of these states, and they all have helicity $\lambda_0 + n/2$. Thus the states with a given helicity form an irreducible representation of the group $U(n)$, characterised by the Young-tableau for n antisymmetric indices,

$$
\left.\begin{array}{c} \boxminus \\ \vdots \\ \boxminus \end{array}\right\} \ n \ \text{rows}
$$

and an R-weight, which can be normalised to be n. $U(n)$ characterises the representations quite independently of whether it is actually a symmetry of the field theory whose spectrum we may be looking at (the actual internal symmetry group is generated by the operators B_r as discussed in section 2, and may be only a subgroup of $U(N)$).

The reason for this is that the representation space of the algebra (3.6) which was the starting point of our discussion of massless particle multiplets will automatically carry a representation of the *automorphism group* of that algebra. This is the group of all linear transformations of q_i and $\bar{q}^i$ into each other such that the algebra itself remains invariant.

In order to construct the largest such group we introduce $2N$ real operators

$$
\begin{aligned}
a_I &\equiv q_i + \bar{q}^i && \text{for } I = i \le N \\
a_I &\equiv \text{i}(q_i - \bar{q}^i) && \text{for } I = N + i,
\end{aligned}
\tag{3.28}
$$

in terms of which the algebra (3.6b) reads

$$
\{a_I, a_J\} = 2\delta_{IJ}.
\tag{3.29}
$$

The largest automorphism group of this is the orthogonal group $O(2N)$ which is generated by the operators

$$
O_{IJ} = \tfrac{1}{4}\text{i}[a_I, a_J].
\tag{3.30}
$$

Two comments: first, O_{IJ} is clearly not a "generator of a symmetry" in the sense of section 2, since it is not bilinear in creation and annihilation operators for particle states (it is quadrilinear), and second, O_{IJ} really generates the covering group spin($2N$) of $O(2N)$, just as the matrices $\tfrac{1}{4}\text{i}[\gamma_\mu, \gamma_\nu]$ generate $Sl(2, c)$ rather than $O(1, 3)$ in the usual Dirac theory.

The 2^N states of our multiplet carry the one and only irreducible representation of the Dirac algebra (3.29). But, of course, all helicities in the multiplet are contained in this. As a representation of

spin($2N$), it is reducible into two 2^{N-1}-dimensional representations, corresponding to the bosonic and fermionic subspaces of our multiplet. The fact that spin($2N$) respects statistics but not helicity is easily understood: its generators are bilinear in fermionic operators – and thus are bosonic – but they are a medley of Q's and $\bar{Q}$'s and thus cannot be expected to preserve helicity. The largest automorphism group which does not mix Q's and $\bar{Q}$'s, and thus will respect helicity, is generated by

$$H_i{}^j = -\tfrac{1}{2}[q_i, \bar{q}^j]$$
(3.31)

and is U(N). Correspondingly, we found that we could classify the states for each helicity in terms of an irreducible representation of U(N).

We now turn to the spectrum generating group for massive representations. First, we observe that the algebra (3.16) has again the group U(N) as the largest automorphism group which leaves the Lorentz indices α and β untouched.

Since a state is created by applying operators $\bar{q}^i{}_{\dot\alpha}$ to a ground state, its Lorentz properties are determined by the symmetry under interchange of $\dot\alpha$-indices, given typically by a Young tableau of the form

The total antisymmetry in index pairs then requires the U(N) properties of this state to be determined by the mirror-image Young tableau

and an R-weight which we can normalise to be the total number of $\bar{q}$'s applied, i.e. the number of squares in the Young tableau.

However, each spin can arise out of several tableaux, e.g. spin $\tfrac{1}{2}$ out of

  ,   etc.,

and thus there may be several U(N) representations for each spin. Therefore the states of any given spin do not generate an irreducible representation of this particular automorphism group. The reason behind this is that there is indeed a larger automorphism group of (3.16) which leaves the spin untouched.

We notice that "spin" is a property of the rotation subgroup rather than of the full Lorentz group itself. Under spatial rotations, the operators q_α and $\bar{q}^{\dot\alpha}$ transform in the same way and we can treat them on the same footing. We define $q_{\alpha I}$ with $I = 1, \dots 2N$ to be

$$q_{\alpha I} \equiv q_{\alpha i} \quad \text{for } I = i \le N$$
$$q_{\alpha I} \equiv \bar{q}^{\dot\alpha i} \quad \text{for } I = i + N \text{ and } \alpha = \dot\alpha \text{ (numerically)}.$$
(3.32)

In terms of these, the algebra becomes

$$\{q_{\alpha I}, q_{\beta J}\} = \varepsilon_{\alpha\beta} C_{IJ} \tag{3.33}$$

with C_{IJ} the matrix elements of the antisymmetric $2N \times 2N$ matrix

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.34}$$

The largest automorphism group of this algebra which leaves the spin indices untouched is the symplectic group Sp(2N). The "symplectic reality condition"

$$(q_{\alpha I})^\dagger = \varepsilon_{\alpha\beta} C_{IJ} q_{\beta J} \tag{3.35}$$

follows from the definitions (3.32), and in order for it to be invariant, the automorphism group must be unitary and thus be the unitary version of Sp(2N), namely USp(2N). This can be seen as follows:

The defining condition for Sp(2N) is that its group elements S leave C invariant:

$$S^T C S = C. \tag{3.36}$$

The condition that (3.35) be invariant, reads

$$(Sq)^\dagger = \varepsilon C(Sq) \qquad \text{or} \qquad S^{\dagger T} q^\dagger = \varepsilon C S q.$$

Multiplying with $S^{-1\dagger T}$ we get

$$q^\dagger = \varepsilon (S^{-1\dagger T} C S) q, \qquad \text{i.e.} \qquad C = S^{-1\dagger T} C S.$$

The unitarity of S follows from comparison with (3.36).
  We conclude that

> USp(2N) *is the spectrum generating group for massive multiplets of N-extended supersymmetry without central charges.*

The generators of this group are

$$D_{IJ} = \tfrac{1}{2} \varepsilon^{\beta\alpha} [q_{\alpha I}, q_{\beta J}]. \tag{3.37}$$

As an example, let us consider the following tables which give the properties of the massive representations with $s_0 = 0$ for $N = 1$ and $N = 2$:

$\underline{N = 1; \ s_0 = 0}$:

spin:	0	$\tfrac{1}{2}$
SU(1)^R:	$1^0 + 1^2$	1^1
USp(2):	2	1

$$\tag{3.38}$$

$N = 2; s_0 = 0$:

spin:	0	$\frac{1}{2}$	1
$\text{SU}(2)^R$:	$1^0 + 3^2 + 1^4$	$2^1 + 2^3$	1^2
$\text{USp}(4)$:	5	4	1

$$\text{(3.39)}$$

$\underline{N = 2;\ s_0 = 0;\ z = m}$:

spin:	0	$\frac{1}{2}$
$\text{USp}(2)$:	$2 + 2$	$1 + 1$.

$$\text{(3.40)}$$

3.6. A representation on fields (the N = 1 chiral multiplet)

By far the most practically important representations of supersymmetry, both "on-shell" and "off-shell", are those on fields. It is they from which supersymmetric field theoretical models are constructed and which have kept theoreticians busy for many years.

In this subsection, the simplest field representation of the simplest supersymmetry algebra, the $N = 1$ chiral multiplet, is constructed explicitly and in detail.

Elements of the Hilbert space of a quantum field theory can be generated by the action of field-valued operators $\phi(x)$ on a translationally invariant vacuum:

$$|x\rangle = \phi(x)\,|0\rangle; \qquad |x, x'\rangle = \phi(x)\,\phi(x')\,|0\rangle\,,$$

etc., and translations of a state are generated by the energy-momentum operator:

$$|x + y\rangle = e^{iy \cdot P}\,|x\rangle; \qquad |x + y, x' + y\rangle = e^{iy \cdot P}\,|x, x'\rangle\,, \tag{3.41}$$

where $y \cdot P \equiv y^\mu P_\mu$. Therefore the displacement of a field takes the form

$$\phi(x + y) = e^{iy \cdot P}\,\phi(x)\,e^{-iy \cdot P} \tag{3.42a}$$

or, differentially,

$$[\phi, P_\mu] = i\,\partial_\mu \phi\,. \tag{3.42b}$$

Because we know the action of P_μ on fields, the structure of the supersymmetry algebra is very similar to that of a Clifford algebra:

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0; \qquad \{Q, \bar{Q}\} = \text{something known}$$

and we can proceed to construct a representation explicitly in "seven easy steps":

Step 1: We choose some complex scalar field $A(x)$ as "ground state" of our representation.

Step 2: We impose the *constraint*

$$[A, \bar{Q}_{\dot{\alpha}}] = 0\,. \tag{3.43}$$

Although this constraint is somewhat reminiscent of the Clifford ground state condition (3.8) – with the roles of Q and $\bar{Q}$ interchanged – it is rather arbitrary at this stage. It defines what is known as the *chiral multiplet*. This name is appropriate since the constraint itself has a chiral index $\dot{\alpha}$.

The following formula, derived from the graded Jacobi identity, the algebra, and eq. (3.42b),

$$\{[A, Q], \bar{Q}\} + \{[A, \bar{Q}], Q\} = [A, \{Q, \bar{Q}\}] = 2i\sigma^{\mu} \partial_{\mu} A, \tag{3.44}$$

implies that A must be *complex*, since otherwise we would have $[A, Q] = 0$ as well as $[A, \bar{Q}] = 0$ and this would mean that A is constant, $\partial_{\mu} A = 0$.

Step 3: We define fields $\psi_{\alpha}(x)$, $F_{\alpha\beta}(x)$ and $X_{\alpha\dot{\beta}}(x)$ by

$$[A, Q_{\alpha}] \equiv 2i\psi_{\alpha}$$

$$\{\psi_{\alpha}, Q_{\beta}\} \equiv -iF_{\alpha\beta}; \qquad \{\psi_{\alpha}, \bar{Q}_{\dot{\beta}}\} \equiv X_{\alpha\dot{\beta}}.$$

Step 4: We enforce the algebra on A. Because of (3.44) and (3.43) this means

$$2i(\sigma^{\mu})_{\alpha\dot{\beta}} \partial_{\mu} A = 2i\{\psi_{\alpha}, \bar{Q}_{\dot{\beta}}\} = 2iX_{\alpha\dot{\beta}}$$

and, from the similar identity for $[A, \{Q, Q\}]$,

$$0 = 2i\{\psi_{\alpha}, Q_{\beta}\} + 2i\{\psi_{\beta}, Q_{\alpha}\} = 2(F_{\alpha\beta} + F_{\beta\alpha}).$$

The solution of the latter equation is

$$F_{\alpha\beta} = \varepsilon_{\alpha\beta} F,$$

with $F(x)$ a complex scalar field.

Step 5: We define fields λ_{α} and $\bar{\chi}_{\dot{\alpha}}$ by

$$[F, Q_{\alpha}] \equiv \lambda_{\alpha}; \qquad [F, \bar{Q}_{\dot{\alpha}}] \equiv \bar{\chi}_{\dot{\alpha}}.$$

Step 6: We enforce the algebra on ψ. The graded Jacobi identities involved are

$$[\{\psi_{\alpha}, Q_{\beta}\}, \bar{Q}_{\dot{\beta}}] + [\{\psi_{\alpha}, \bar{Q}_{\dot{\beta}}\}, Q_{\beta}] = [\psi_{\alpha}, \{Q_{\beta}, \bar{Q}_{\dot{\beta}}\}] \tag{3.45}$$

and a similar one for $[\{\psi, Q\}, Q]$. The resulting conditions are

$$-i\varepsilon_{\alpha\beta}\bar{\chi}_{\dot{\beta}} + 2i(\sigma^{\mu})_{\alpha\dot{\beta}} \partial_{\mu}\psi_{\beta} = 2i(\sigma^{\mu})_{\beta\dot{\beta}} \partial_{\mu}\psi_{\alpha}, \qquad \varepsilon_{\alpha\beta}\lambda_{\gamma} + \varepsilon_{\alpha\gamma}\lambda_{\beta} = 0$$

and have only one solution:

$$\bar{\chi}_{\dot{\alpha}} = 2 \partial_{\mu}\psi^{\beta}(\sigma^{\mu})_{\beta\dot{\alpha}}; \qquad \lambda_{\alpha} = 0.$$

This is shown by contracting indices α and β with the help of $\varepsilon^{\alpha\beta}$.

Step 7: We have to check the remaining conditions

$$[\psi, \{\bar{Q}, \bar{Q})] = [F, \{Q, Q\}] = [F, \{\bar{Q}, \bar{Q}\}] = 0$$

$$[F, \{Q, \bar{Q}\}] = 2i\sigma^\mu \, \partial_\mu F,$$

which are found to be satisfied.

We have thus constructed a representation of the $N = 1$ supersymmetry algebra on a multiplet ϕ of fields

$$\phi = (A; \psi; F).$$ (3.46)

The representation is in terms of the (anti)commutators

$$[A, Q_\alpha] = 2i\psi_\alpha \qquad\qquad [A, \bar{Q}_{\dot\alpha}] = 0$$

$$\{\psi_\alpha, Q_\beta\} = -i\varepsilon_{\alpha\beta}F \qquad \{\psi_\alpha, \bar{Q}_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta} \, \partial_\mu A$$ (3.47)

$$[F, Q_\alpha] = 0 \qquad\qquad [F, \bar{Q}_{\dot\alpha}] = 2 \, \partial_\mu\psi^\beta (\sigma^\mu)_{\beta\dot\alpha} \, .$$

Usually, one introduces *anticommuting spinor parameters* ζ^α and

$$\bar{\zeta}^{\dot\alpha} \equiv (\zeta^\alpha)^*$$ (3.48)

which are defined to anticommute with everything fermionic (including themselves) and to commute with everything bosonic (including, of course, ordinary c-numbers). They are something like "fermionic numbers". It should be mentioned that there ought to be no mathematical inconsistency in this: spinorial quantum fields $\zeta(y)$, e.g., have these properties if their coordinates y^μ are space-like to each other and to the region where we do our physics [33].

With the help of the spinor parameters, we can define infinitesimal *supersymmetry variations* of a field ϕ by

$$\delta\phi \equiv -i[\phi, \zeta Q + \bar{Q}\bar{\zeta}] \, ,$$ (3.49)

in analogy with eq. (3.42b) (see the Technical Appendix for proper definitions of implied summation over spinor indices and Hermitian conjugation).

For our multiplet, the transformation laws are

$$\delta A = 2\zeta\psi$$

$$\delta\psi = -\zeta F - i \, \partial_\mu A \, \sigma^\mu \bar{\zeta}$$ (3.50)

$$\delta F = -2i \, \partial_\mu\psi \, \sigma^\mu \bar{\zeta}$$

and the whole algebra can be written as

$$[\delta_1, \delta_2]\phi = 2i(\zeta_1\sigma^\mu \bar{\zeta}_2 - \zeta_2\sigma^\mu \bar{\zeta}_1) \, \partial_\mu\phi \, .$$ (3.51)

Let us count degrees of freedom for the multiplet ϕ. A "degree of freedom" is an unconstrained single real field. There are total of four bosonic ones: Re A, Im A, Re F and Im F, and four fermionic ones: Re ψ_1, Im ψ_1, Re ψ_2 and Im ψ_2. The multiplet has thus $4+4$ degrees of freedom; this is the *smallest possible number* in four space–time dimensions, since any multiplet must contain a spinor, and spinors have at least two complex or four real components (Weyl and Majorana spinors, respectively). The multiplet is therefore *irreducible*.

The algebra itself, most obviously in the form (3.51), prohibits a trivial representation $\delta\phi = 0$, since this would imply $\phi = $ const. Thus,

the only trivial representation of supersymmetry are constant fields.

The field multiplet ϕ, which we have derived in this subsection, is called by various names: "chiral multiplet" (which I prefer and shall use), "scalar multiplet" (the name given it by Wess and Zumino in their first paper [71]) and "Wess–Zumino multiplet".

If we had, in step 2, set a constraint $[A, Q] = 0$ instead of $[A, \bar{Q}] = 0$, we would have got an *anti-chiral multiplet* $\bar{\phi}$. Such a multiplet can be constructed from ϕ by Hermitian conjugation, because indeed $[A^\dagger, Q] = 0$. The multiplet has components

$$\bar{\phi} = (A^\dagger; \bar{\psi}; F^\dagger) \tag{3.52}$$

and transformation laws

$$\delta A^\dagger = 2\bar{\psi}\bar{\zeta}$$

$$\delta\bar{\psi} = -F^\dagger\bar{\zeta} + i\zeta\sigma^\mu\,\partial_\mu A^\dagger \tag{3.53}$$

$$\delta F^\dagger = 2i\zeta\sigma^\mu\,\partial_\mu\bar{\psi}.$$

The anti-chiral multiplet contains the same $4+4$ real field components as the chiral one, they are just arranged differently.

Further chiral multiplets can be derived by giving every field in ϕ or $\bar{\phi}$ an *additional Lorentz index*. Such multiplets are not irreducible. Care has to be taken when adding a spinor index because this inverts the statistics of the fields. Indeed, eqs. (3.50) and (3.53) can be used to obtain the transformation laws for multiplets with inverse statistics only after all ζ have been commuted to the right of the fields. I conclude this section by pointing out that the principle of the "seven easy steps" can be applied to much more complicated cases, e.g. in supergravity where the algebra is more complicated and involves various field dependent gauge transformations. The steps are then not so easy.

4. $N = 1$ multiplets and tensor calculus

In this section, the real general multiplet is introduced [71] and more properties of the chiral multiplet of $N = 1$ supersymmetry are discussed. We shall see how we can combine multiplets and how we can construct "invariant" scalar products from any two of them [73].

4.1. Four-component notation

From now on whenever component multiplets are used, I drop chiral notation (dotted and undotted two-spinor indices) and adopt the usual four-component notation of relativistic spinor physics. The advantages and disadvantages of either notation weigh about equal (chiral notation is somewhat easier to manage in calculations, four-spinors allow a more concise presentation of results), and since each is widely used in articles on supersymmetric theories, they must both be introduced. The basic formulas relating the two are contained in the Technical Appendix, section A.5.

For any N-extended superalgebra, we can define Majorana spinors from our chiral $Q_{\alpha i}$,

$$Q_i = \begin{bmatrix} Q_{\alpha i} \\ \bar{Q}^{\dot{\alpha} i} \end{bmatrix}; \qquad \bar{Q}_i = (Q_i^\alpha, \bar{Q}_{\dot{\alpha}}^i) \,. \tag{4.1}$$

In this four-spinor notation, the supersymmetry algebra where it involves Q becomes

$$\{Q_i, \bar{Q}_j\} = 2(\delta_{ij}\gamma^\mu P_\mu + \mathrm{i}\,\mathrm{Im}\,Z_{ij} + \mathrm{i}\gamma_5\,\mathrm{Re}\,Z_{ij})$$

$$[Q, P_\mu] = 0; \qquad [Q, M_{\mu\nu}] = \tfrac{1}{2}\sigma_{\mu\nu}Q; \qquad [Q, R] = \mathrm{i}\gamma_5 Q \,. \tag{4.2}$$

We also have, with a Majorana spinor ζ^i defined from $\zeta^{\alpha i}$:

$$\zeta^{\alpha i}Q_{\alpha i} + \bar{Q}_{\dot{\alpha}}^i \bar{\zeta}_i^{\dot{\alpha}} = \bar{\zeta}Q; \qquad \delta\phi = -\mathrm{i}[\phi, \bar{\zeta}Q]$$

$$[\delta_1, \delta_2]\phi = 2\mathrm{i}\bar{\zeta}_1\gamma^\mu\zeta_2\,\partial_\mu\phi + 2\mathrm{i}\bar{\zeta}_1^i[\phi, \mathrm{Im}\,Z_{ij} + \gamma_5\,\mathrm{Re}\,Z_{ij}]\zeta_2^j \,. \tag{4.3}$$

We note that whereas some symplectic subgroup of $U(N)$ was an automorphism group of the algebra in the chiral notation, eq. (2.37), our definition of the Majorana charges mixes upper and lower $U(N)$-indices, and we are left with some orthogonal subgroup of the original automorphism group as the new, smaller automorphism group of (4.2).

A real (Majorana) form of the $N = 1$ chiral multiplet is obtained in the following way:

We call the real and imaginary parts of the complex fields A and $F^\dagger$ in the chiral multiplet by the names A, B and F, G and hope that there is no confusion over the symbols A and F. We then construct a Majorana spinor ψ from the chiral spinor ψ_α and its conjugate $\bar{\psi}_{\dot{\alpha}}$. The transformation laws for the resulting multiplet

$$\phi = (A, B; \psi; F, G) \tag{4.4}$$

can be calculated from the complex version (3.50) in a straightforward manner ($\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$):

$$\delta A = \bar{\zeta}\psi; \qquad \delta B = \bar{\zeta}\gamma_5\psi \,;$$

$$\delta\psi = -(F + \gamma_5 G)\zeta - \mathrm{i}\,\partial\!\!\!/(A + \gamma_5 B)\zeta \,; \tag{4.5}$$

$$\delta F = \mathrm{i}\,\bar{\zeta}\partial\!\!\!/\psi; \qquad \delta G = \mathrm{i}\,\bar{\zeta}\gamma_5\partial\!\!\!/\psi \,.$$

The parameter ζ is a Majorana spinor constructed from ζ^α and $\bar{\zeta}^{\dot{\alpha}}$. The presence of γ_5's in (4.5) gives clear parity assignments: A and F are scalars, B and G are pseudo-scalars.

The defining property of the chiral multiplet, namely the absence of $\bar{\zeta}^{\dot{\alpha}}$ from the variation of the complex A-field, manifests itself now in that the spinor in δB is just γ_5 times the one in δA.

We could have derived the transformations (4.5) directly in a process similar to the "seven easy steps" of section 3, with step 1 being the choice of a scalar–pseudoscalar pair (A, B), step 2 being the constraint that δB is just like δA with an additional γ_5, and with the algebra to be enforced being that of eq. (4.2) with $N = 1$, and therefore without central charges.

Since ϕ and $\bar{\phi}$ have the same real-field content, there is no need to introduce a separate real form of the anti-chiral multiplet.

4.2. The general multiplet

What would have happened to the construction of the chiral multiplet without the chirality constraint $[A, \bar{Q}_{\dot{\alpha}}] = 0$? The basic principle behind the "seven easy steps" was that the anticommutator $\{Q, Q\}$ is "something known" which of course still is true. Therefore, we can derive a multiplet from a general "ground state" field $C(x)$. This will take considerably more than seven steps, and results in a larger multiplet

$$V = (C; \chi; M, N, A_\mu; \lambda; D),$$ (4.6)

with transformation laws which in four-component notation read

$$\delta C = \bar{\zeta}\gamma_5\chi$$

$$\delta\chi = (M + \gamma_5 N)\zeta - i\gamma^\mu (A_\mu + \gamma_5 \partial_\mu C)\zeta$$

$$\delta M = \bar{\zeta}(\lambda - i\,\partial\!\!\!/\chi)$$

$$\delta N = \bar{\zeta}\gamma_5(\lambda - i\,\partial\!\!\!/\chi)$$ (4.7)

$$\delta A_\mu = i\,\bar{\zeta}\gamma_\mu\lambda + \bar{\zeta}\partial_\mu\chi$$

$$\delta\lambda = -i\,\sigma^{\mu\nu}\zeta\,\partial_\mu A_\nu - \gamma_5\zeta D$$

$$\delta D = -i\,\bar{\zeta}\,\partial\!\!\!/\gamma_5\lambda.$$

The scalar M, the pseudoscalars C, N and D and the vector A_μ are a-priori complex and the spinors χ and λ are Dirac spinors. In contrast to the chiral multiplet, however, it is possible to impose a reality condition

$$V = V^\dagger$$ (4.8)

which is defined to mean that all components are real or Majorana. This is known as the *real general multiplet*, "general" because there is no constraint imposed on the transformation laws other than that they must represent the algebra. This multiplet has $8 + 8$ field components.

4.3. Reducibility and submultiplets

Closer inspection of the real general multiplet reveals that it is not an irreducible representation of

the supersymmetry algebra. Instead, we find that the fields λ, D and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ transform entirely among themselves, and thus form a *submultiplet* dV with components

$$dV = (\lambda; F_{\mu\nu}, D).\tag{4.9}$$

These are $4+4$ real field components, and we call the multiplet by the name *curl multiplet*. This submultiplet is irreducible, but V itself is not. The transformation laws

$$\delta\lambda = -\tfrac{1}{2}i\,\sigma^{\mu\nu}\zeta F_{\mu\nu} - \gamma_5\zeta D$$
$$\delta F_{\mu\nu} = -i\,\bar{\zeta}(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu)\,\lambda \tag{4.10}$$
$$\delta D = -i\,\bar{\zeta}\,\slashed{\partial}\gamma_5\lambda$$

represent the algebra, provided that the constraint

$$\partial_{[\kappa}F_{\mu\nu]} = 0 \tag{4.11}$$

holds. This is the case for $F_{\mu\nu}$ being a curl.

A different submultiplet of (4.6) is the chiral multiplet

$$\partial V = (M, N; \lambda - i\,\slashed{\partial}\chi; \partial^\mu A_\mu, D + \Box C).\tag{4.12}$$

Both dV and ∂V are submultiplets of V, but their complements are not: the transformation law for χ, e.g., contains the transverse part of A_μ as well as both M and N. This property of being *reducible but not fully reducible* is very common for multiplets of supersymmetry.

Since the dV and ∂V are submultiplets, we can *constrain* V to not contain one or the other:

$$dV = 0 \qquad\text{or}\qquad \partial V = 0.\tag{4.13a, b}$$

In these cases the remaining fields now do transform solely into each other. For $dV = 0$, the surviving components of V can be arranged into a chiral multiplet

$$\phi = (A, C; \chi; -M, -N),\tag{4.14}$$

with the scalar A defined by the solution $A_\mu = \partial_\mu A$ of $F_{\mu\nu} = 0$. For $\partial V = 0$, the surviving components form a new type of multiplet which contains C, χ and a divergence-free vector A_μ:

$$L = (C; \chi; A_\mu).\tag{4.15}$$

This is called the *linear multiplet*; it is irreducible and has transformation laws

$$\delta C = \bar{\zeta}\gamma_5\chi$$
$$\delta\chi = -i\,\gamma^\mu\,(A_\mu + \gamma_5\,\partial_\mu C)\,\zeta \tag{4.16}$$
$$\delta A_\mu = i\,\bar{\zeta}\sigma_{\mu\nu}\,\partial^\nu\chi$$

which represent the supersymmetry algebra, provided that the constraint

$$\partial_\mu A^\mu = 0 \tag{4.17}$$

holds.

The structure of a not fully reducible multiplet can be summarized symbolically as

Z contains X and Y

δX contains only X, but

δY contains X and Y.

In our cases, we had $Z = V$ and either $X = dV$ and $Y = \phi$, or $X = \partial V$ and $Y = L$.

4.4. The product multiplets

One can build a chiral multiplet ϕ_3 as the product of two others ϕ_1 and ϕ_2: with $[A_1, \bar{Q}] = 0$ and $[A_2, \bar{Q}] = 0$, we also have $[A_1 A_2, \bar{Q}] = 0$, all for complex A's. In real components this means that the spinor in $\delta(A_1 B_2 + B_1 A_2)$ is γ_5 times that in $\delta(A_1 A_2 - B_1 B_2)$. Explicit calculation shows that the components of the product of two chiral multiplets

$$\phi_3 = \phi_1 \cdot \phi_2 \tag{4.18}$$

are

$$A_3 = A_1 A_2 - B_1 B_2$$

$$B_3 = B_1 A_2 + A_1 B_2$$

$$\psi_3 = (A_1 - \gamma_5 B_1)\psi_2 + (A_2 - \gamma_5 B_2)\psi_1 \tag{4.19}$$

$$F_3 = F_1 A_2 + A_1 F_2 + B_1 G_2 + G_1 B_2 + \bar{\psi}_1 \psi_2$$

$$G_3 = G_1 A_2 + A_1 G_2 - B_1 F_2 - F_1 B_2 - \bar{\psi}_1 \gamma_5 \psi_2 .$$

The product of three multiplets is associative, and thus the product of any number of them is well-defined.

Two other products of chiral multiplets can be defined. One is denoted by

$$V = \phi_1 \times \phi_2 \tag{4.20}$$

and is neither chiral nor anti-chiral but rather a general real multiplet with components

$$C \;\; = A_1 A_2 + B_1 B_2$$

$$\chi \;\; = (B_1 - \gamma_5 A_1)\psi_2 + (B_2 - \gamma_5 A_2)\psi_1$$

$$M \;\; = -G_1 A_2 - F_1 B_2 - A_1 G_2 - B_1 F_2$$

$$N \;\; = F_1 A_2 - G_1 B_2 + A_1 F_2 - B_1 G_2 \tag{4.21}$$

$$A_\mu = B_1 \overleftrightarrow{\partial}_\mu A_2 + B_2 \overleftrightarrow{\partial}_\mu A_1 + i \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2$$

$$\lambda \;\; = -(G_1 + \gamma_5 F_1 - i \,\partial\!\!\!/ B_1 - i \gamma_5 \,\partial\!\!\!/ A_1)\psi_2 + (1 \leftrightarrow 2)$$

$$D \;\; = -2F_1 F_2 - 2G_1 G_2 - 2 \,\partial_\mu A_1 \,\partial^\mu A_2 - 2 \,\partial_\mu B_1 \,\partial^\mu B_2 - i \bar{\psi}_1 \overleftrightarrow{\partial\!\!\!/} \psi_2 .$$

Like $\phi_1 \cdot \phi_2$, this is symmetric under interchange of ϕ_1 and ϕ_2. The third type of product is antisymmetric, is denoted by

$$V = \phi_1 \wedge \phi_2 , \tag{4.22}$$

is again a general real multiplet and has components

$$C = A_1 B_2 - B_1 A_2$$

$$\chi = (A_1 + \gamma_5 B_1)\psi_2 - (A_2 + \gamma_5 B_2)\psi_1$$

$$M = F_1 A_2 - G_1 B_2 - A_1 F_2 + B_1 G_2$$

$$N = F_1 B_2 + G_1 A_2 - B_1 F_2 - A_1 G_2 \tag{4.23}$$

$$A_\mu = A_1 \overset{\leftrightarrow}{\partial}_\mu A_2 + B_1 \overset{\leftrightarrow}{\partial}_\mu B_2 - i\bar\psi_1 \gamma_\mu \psi_2$$

$$\lambda = (F_1 - \gamma_5 G_1 + i \overset{\leftrightarrow}{\partial} A_1 + i \overset{\leftrightarrow}{\partial}\gamma_5 B_1)\psi_2 - (1 \leftrightarrow 2)$$

$$D = -2G_1 F_2 + 2F_1 G_2 + 2\, \partial_\mu B_1\, \partial^\mu A_2 - 2\, \partial_\mu A_1\, \partial^\mu B_2 + i\bar\psi_1 \gamma_5 \overset{\leftrightarrow}{\partial}\psi_2 \,.$$

Finally, we can also define a product for two real general multiplets

$$V_3 = V_1 \cdot V_2 \tag{4.24}$$

which is again a real general multiplet, is symmetric in V_1 and V_2 and has components

$$C_3 = C_1 C_2$$

$$\chi_3 = C_1 \chi_2 + \chi_1 C_2$$

$$M_3 = C_1 M_2 + M_1 C_2 - \tfrac{1}{2}\bar\chi_1 \gamma_5 \chi_2$$

$$N_3 = C_1 N_2 + N_1 C_2 - \tfrac{1}{2}\bar\chi_1 \chi_2 \tag{4.25}$$

$$A_3^\mu = C_1 A_2^\mu + A_1^\mu C_2 + \tfrac{1}{2}i\bar\chi_1 \gamma^\mu \gamma_5 \chi_2$$

$$\lambda_3 = C_1 \lambda_2 + \tfrac{1}{2}(N_1 + \gamma_5 M_1 + i\overset{\leftrightarrow}{\partial} C_1 - i A_1 \gamma_5)\chi_2 + (1 \leftrightarrow 2)$$

$$D_3 = C_1 D_2 + D_1 C_2 - M_1 M_2 - N_1 N_2 - \partial_\mu C_1\, \partial^\mu C_2 - A_{1\mu} A_2^\mu + \bar\lambda_1 \chi_2 + \bar\chi_1 \lambda_2 - \tfrac{1}{2}i\bar\chi_1 \overset{\leftrightarrow}{\partial}\chi_2 \,.$$

4.5. The "kinetic multiplet"

There are various ways of seeing that another chiral multiplet is contained in ϕ: we can take the adjoint of (3.47) and observe that $[F^\dagger, \bar Q] = 0$; or we can notice that $\delta F^\dagger$ does not contain $\bar\zeta^\alpha$ in (3.53); or we can see that in (4.5) the spinor in δG is just γ_5 times the one in δF. In any case, the real fields F and G are the start of another chiral multiplet, the *kinetic multiplet*, denoted by $\mathsf{T}\phi$. Its components are

$$\mathsf{T}\phi = (F, G; i\overset{\leftrightarrow}{\partial}\psi; -\Box A, -\Box B) \,. \tag{4.26}$$

We can, of course, go on and do this again; but by inspection we see that

$$\mathsf{T}\mathsf{T}\phi = -\Box\phi \,, \tag{4.27}$$

which is why the multiplet is called kinetic: T is something like the generalisation of the massless Dirac

operator:

$$\mathsf{T} \leftrightarrow i\partial\!\!\!/.$$

4.6. Contragradient multiplet and invariants

We now try to construct a multiplet which is contragradient to the chiral multiplet. This means that there should be a bilinear "inner product" of the form

$$L = Af + Bg + \bar{\psi}\chi + Fa + Gb \tag{4.28}$$

which is invariant. The new fields a, b, χ, f and g would be those of the *contragradient multiplet*. We do, however, already know that only constants can be invariant (see the discussion following eq. (3.51)), so that we can at most expect to be able to demand that L be a *density*, which transforms into a divergence,

$$\delta L = \partial^{\mu}\kappa_{\mu}. \tag{4.29}$$

If L is a density, then $\int d^4x\, L$ is invariant (and constant). Thus we can hope to construct Lagrangians from our densities. This behaviour is similar to that of fields (including Lagrangians) under translations – a similarity which makes sense since two successive supersymmetry transformations generate a translation. Using (4.5) we get for δL

$$\delta L = A\,\delta f + B\,\delta g + \bar{\psi}\,\delta\chi + F\,\delta a + G\,\delta b + f\bar{\zeta}\psi + g\bar{\zeta}\gamma_5\psi - \bar{\zeta}(F + \gamma_5 G)\chi$$
$$+ i\,\bar{\zeta}\partial\!\!\!/(A - \gamma_5 B)\chi + i\,\bar{\zeta}\partial\!\!\!/\psi a + i\,\bar{\zeta}\gamma_5\partial\!\!\!/\psi\, b.$$

We partially integrate the last three terms,

$$= \cdots - i\,\bar{\zeta}(A + \gamma_5 B)\partial\!\!\!/\chi - i\,\bar{\zeta}\partial\!\!\!/(a - \gamma_5 b)\psi + \partial^{\mu}[i\,\bar{\zeta}(A + \gamma_5 B)\gamma_{\mu}\chi + i\,\bar{\zeta}(a + \gamma_5 b)\gamma_{\mu}\psi],$$

and we demand that δL should just be the divergence in the last line. We now compare coefficients for A, B, $\bar{\psi}$, F and G and get transformation laws for the fields of the contragradient multiplet:

$$\delta a = \bar{\zeta}\chi; \qquad \delta b = \bar{\zeta}\gamma_5\chi$$

$$\delta\chi = -(f + \gamma_5 g)\zeta - i\,\partial\!\!\!/(a + \gamma_5 b)\zeta$$

$$\delta f = i\,\bar{\zeta}\partial\!\!\!/\chi; \qquad \delta g = i\,\bar{\zeta}\gamma_5\partial\!\!\!/\chi.$$

These are again those of a chiral multiplet,

$$\phi' = (a, b; \chi; f, g),$$

which means that the chiral multiplet is *self-contragradient*, and that we can construct a density from any two chiral multiplets. We notice that the density (4.28) is just the F-component of the product

$\phi \cdot \phi'$:

$$L = (\phi \cdot \phi')_F . \tag{4.30}$$

This is quite consistent since the transformation laws imply that

any F-component of a chiral multiplet is a scalar density.

In a similar fashion, we construct a multiplet which is contragradient to the real general multiplet. We start with the ansatz

$$L = CD' + \bar{\chi}\lambda' - MM' - NN' - A^\mu A'_\mu + \bar{\lambda}\chi' + DC' \tag{4.31}$$

and from the known transformation laws of the fields of the general multiplet, we derive those of the new fields by comparing coefficients. The result is

$$\delta C' = \bar{\zeta}\gamma_5\chi'$$

$$\delta\chi' = (M' + \gamma_5 N')\zeta - i\,\gamma^\mu(A'_\mu + \gamma_5\,\partial_\mu C')\,\zeta$$

$$\delta M' = \bar{\zeta}\lambda'$$

$$\delta N' = \bar{\zeta}\gamma_5\lambda' \tag{4.32}$$

$$\delta A'_\mu = i\,\bar{\zeta}\gamma_\mu\lambda' + i\,\bar{\zeta}\sigma_{\mu\nu}\,\partial^\nu\chi'$$

$$\delta\lambda' = -(\partial^\mu A'_\mu + \gamma_5 D')\zeta - i\,\partial\!\!\!/(M' + \gamma_5 N')\zeta$$

$$\delta D' = -i\,\bar{\zeta}\partial\!\!\!/\gamma_5\lambda' .$$

These transformations are very closely related to those of the real general multiplet. Indeed, certain combinations of these fields form a real general multiplet:

$$V' = (C'; \chi'; M', N', A'_\mu; \lambda' + i\,\partial\!\!\!/\chi'; D' - \Box C') . \tag{4.33}$$

Comparison with eq. (4.12), or simple inspection of the transformation laws (4.32), will show that this shifted form of the general multiplet is particularly suited for detecting the chiral submultiplet $\partial V'$ and the linear multiplet L to which V' reduces when $\partial V' = 0$. The other form of V was more suited to make the curl submultiplet dV apparent.

Since

any D-component of a general multiplet is a scalar density,

it is not surprising to see that the invariant (4.31) can also be written as

$$L = (V \cdot V')_D + 4\text{-div} . \tag{4.34}$$

4.7. Dimensions

If we define the (mass) dimension Δ of a chiral multiplet as that of its A-component, we find that

$$\dim A = \dim B = \Delta \equiv \dim \phi$$
$$\dim \psi = \Delta + \tfrac{1}{2} \tag{4.35}$$
$$\dim F = \dim G = \Delta + 1$$

because the parameter ζ must have dimension $-\tfrac{1}{2}$ so that $i\bar{\zeta}_1\gamma^\mu\zeta_2$ is a length. The assignments (4.35) then follow from the transformation laws (4.5). Furthermore,

$$\dim(\phi_1 \cdot \phi_2) = \Delta_1 + \Delta_2$$
$$\dim \mathsf{T}\phi = \Delta + 1$$
$$\dim(\phi \cdot \mathsf{T}\phi)_F = 2\Delta + 2 \tag{4.36}$$
$$\dim(\phi^n)_F = n\Delta + 1 \;.$$

5. The Wess–Zumino model

The Wess–Zumino model consists of a single chiral multiplet ϕ in renormalisable self-interaction [72].

5.1. The Lagrangian and the equations of motion

If we want to construct a Lagrangian which describes the motion of fields in time, it must contain derivatives. If the maximal number of derivatives is to be two (for the boson fields), we must involve exactly one of the T-operations of subsection 4.5. This, together with the demand that $\dim L = 4$, determines the dimension of the multiplet to be $\Delta = 1$. The most general renormalisable Lagrangian for a single chiral multiplet (i.e. one with no coupling constants of negative dimension) is then the *Wess–Zumino Lagrangian*:

$$L = \left(\tfrac{1}{2}\phi \cdot \mathsf{T}\phi - \frac{m}{2}\,\phi \cdot \phi - \frac{g}{3}\,\phi \cdot \phi \cdot \phi\right)_F = L_0 + L_m + L_g \tag{5.1}$$

(a further possible term $(\lambda\phi)_F$ can be eliminated by field redefinitions, more details of this can be found in section 6). Using many of the tensor calculus results of the previous section, we can spell this out in components:

$$L_0 = \tfrac{1}{2}(\partial_\mu A \, \partial^\mu A + \partial_\mu B \, \partial^\mu B + i\,\bar{\psi}\slashed{\partial}\psi + F^2 + G^2) + \text{4-div}$$
$$L_m = -m(AF + BG + \tfrac{1}{2}\bar{\psi}\psi) \tag{5.2}$$
$$L_g = -g[(A^2 - B^2)F + 2ABG + \bar{\psi}(A - \gamma_5 B)\psi] \;.$$

The term "4-div" in L_0 stands for the divergence which arose out of the partial integration which replaced $-\frac{1}{2}(A \,\Box A + B \,\Box B)$ by the standard $\frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2$.

From this Lagrangian, the following Euler–Lagrange *equations of motion* can be derived,

$$F = mA + g(A^2 - B^2)$$

$$G = mB + 2gAB$$

$$\mathrm{i}\,\partial\!\!\!/\psi = m\psi + 2g(A - \gamma_5 B)\psi \qquad\qquad (5.3\mathrm{a})$$

$$-\Box A = mF + 2g(AF + BG + \tfrac{1}{2}\bar\psi\psi)$$

$$-\Box B = mG + 2g(AG - BF - \tfrac{1}{2}\bar\psi\gamma_5\psi),$$

which can be written most elegantly in multiplet form

$$\mathsf{T}\phi = m\phi + g\phi \cdot \phi. \qquad\qquad (5.3\mathrm{b})$$

5.2. Auxiliary fields

The Lagrangian and the equations of motion in their present form may not be very illuminating as far as the "physics" of the model is concerned. Notice, however, that the field equations for F and G are purely algebraic. These fields can be eliminated from the Lagrangian and from the equations of motion by use of their own field equations. Such fields whose equations of motion do not describe propagation in time and space, are called *auxiliary fields*. The result of their elimination is in our case the "on-shell" Lagrangian*

$$L = \tfrac{1}{2}(\partial_\mu A \,\partial^\mu A - m^2 A^2) + \tfrac{1}{2}(\partial_\mu B \,\partial^\mu B - m^2 B^2) + \tfrac{1}{2}\bar\psi(\mathrm{i}\,\partial\!\!\!/ - m)\psi$$

$$- mgA(A^2 + B^2) - g\bar\psi(A - \gamma_5 B)\psi - \frac{g^2}{2}(A^2 + B^2)^2 \qquad\qquad (5.4)$$

and the "on-shell" equations of motion

$$(\Box + m^2)A = -mg(3A^2 + B^2) - 2g^2 A(A^2 + B^2) - g\bar\psi\psi$$

$$(\Box + m^2)B = -2mgAB - 2g^2 B(A^2 + B^2) + g\bar\psi\gamma_5\psi \qquad\qquad (5.5)$$

$$(\mathrm{i}\,\partial\!\!\!/ - m)\psi = 2g(A - \gamma_5 B)\psi.$$

What does the Lagrangian (5.4) know of the supersymmetry of the model? Actually, it is not the most general renormalisable Lagrangian for the fields involved. All seven possible parity-invariant interaction terms

$$A^3, AB^2, A^4, B^4, A^2 B^2, A\bar\psi\psi \quad \text{and} \quad B\bar\psi\gamma_5\psi,$$

---

* The somewhat idiosyncratic use of "on-shell" and "off-shell" in supersymmetry is discussed below (subsection 5.4).

and the three possible mass terms are present, but all masses are the same (O'Raifeartaigh's theorem again!) and all seven couplings are fixed in terms of the mass and a single coupling constant g. This reflects the fact that the Lagrangian, even after the fields F and G have been eliminated, still transforms as a density under the "on-shell" transformations

$$\delta A = \bar{\zeta}\psi$$

$$\delta B = \bar{\zeta}\gamma_5\psi \qquad (5.6)$$

$$\delta\psi = -[i\slashed{\partial} + m + g(A + \gamma_5 B)](A + \gamma_5 B)\zeta$$

which are derived from (4.5) by elimination of the fields F and G, using their own equations of motion. There is no need to employ the dynamical equations of motion (5.5) to show that (5.4) transforms as a density under (5.6).

5.3. Comments on renormalisation

Particularly from this last discussion, we see that it is actually the particular relationship between the coupling constants of the Lagrangian (5.4) that constitutes the supersymmetry of the model. The question is now whether this relationship is stable under *renormalisation*. This has been found to be so [72, 39], since there actually exists a regularisation which preserves supersymmetry for the model (Pauli–Villars).

More surprisingly, the independent renormalisations of m and g are not needed since all contributions to the quadratically and the linearly divergent terms *cancel*, due to fermionic and bosonic loops contributing with opposite signs. There is only a single, logarithmically divergent infinity which can be absorbed into a wave-function renormalisation which is common to the whole multiplet, i.e. the same for all the fields A, B and ψ.

This absence of infinities which are not prohibited by the impossibility to write down a supersymmetric counterterm (the counterterm $\delta m\,(\phi \cdot \phi)_F$ would, e.g., be perfectly supersymmetric) is called "miraculous cancellations". They are now understood in terms of *non-renormalisation theorems* derived in superspace [30, 31].

The "miraculous cancellations" require the presence, in a very controlled way, of fields whose loops contribute consistently with opposite signs, as bosons and fermions do. A "well-controlled relationship between fermions and bosons" is a supersymmetry.

5.4. On-shell and off-shell representations

One of the important open problems in the field of supersymmetry is to find "off-shell" formulations for the higher-N extended models or to prove that they don't exist. Using the Wess–Zumino model as an example, it is straightforward to explain what is meant by off-shell and on-shell representations.

The transformation laws (4.5) represent the supersymmetry algebra on the $4 + 4$ field components of the chiral multiplet, unconditionally and independently of the dynamics, i.e. of a Lagrangian. Each of the three terms in the Wess–Zumino Lagrangian (5.1) separately transforms as a density under those transformations. This is called *off-shell supersymmetry*.

The complete set of field equations (5.2) is again a multiplet, in this case actually a chiral multiplet. The structure of these field equations, however, separates them into two classes, algebraic equations for

some fields (which are thus auxiliary), and wave equations for the others (which thus describe dynamical degrees of freedom).

The elimination of auxiliary fields takes place by enforcing their field equations on the Lagrangian and on the transformation rules. This procedure is not itself supersymmetric: since all field equations sit in an irreducible multiplet, we can retain the supersymmetry only by taking all fields "on-shell", i.e. subject to their equations of motion. What does this mean in practice?

The *on-shell Lagrangian* (5.4) is still a density under the *on-shell transformations* (5.6). These transformations, however, have become dependent on the model (they depend on m and g), and there is no part of the Lagrangian which separately transforms as a density under them.

In gauge theories and supergravity this leads to an inseparable marriage between background and matter which is tolerable at most for the maximally extended theories ($N = 4$ and $N = 8$) where there is no matter and where everything is a supersymmetric version of gauge or gravitational background.

The most serious drawback of "on-shell formulations" appears if we calculate the commutator of two of the transformations (5.6). The result is $2i\bar\zeta_1 \mathcal{A}\zeta_2$ on A and B, as it should be, but

$$[\delta_1, \delta_2]\psi = 2i\bar\zeta_1 \mathcal{A}\zeta_2 \, \psi - \gamma^\mu (i\mathcal{A} - m - 2g(A - \gamma_5 B))\psi\bar\zeta_1\gamma_\mu\zeta_2 \tag{5.7}$$

on ψ. This means that

   the on-shell algebra closes only if the equations of motion hold,

which set the last term in (5.7) to zero. This situation could in principle be disastrous for quantum corrections: there the fields must be taken off-shell, away from their classical paths through configuration space. What this does to the on-shell supersymmetric theories is quite unclear. Obviously, if there exists – even unknown to us – some unique off-shell version, there should be no problem. But what happens if the theory were intrinsically only on-shell supersymmetric (as the $N = 4$ and $N = 8$ theories may well be) or if there were several competing off-shell versions (as there are for $N = 1$ supergravity), is unclear to date.

There are several ways to understand why on- and off-shell representations are different and why auxiliary fields appear in supersymmetric theories. At the root of the problem lies the difference between the vector space for the off-shell representations, fields over x^μ, and that for the on-shell representations, the Cauchy data for the field equations. In both cases the fermions = bosons rule must hold. For different spins, however, there is a different relationship between Cauchy data and fields. While, e.g., a real scalar field $A(x)$ describes one neutral scalar particle, the Dirac field $\psi(x)$ has eight real components, but describes only the four states of a charged spin-$\frac{1}{2}$ particle. Thus, in going from the fields to the states, we have lost some dimensions of our representation space, but differently so for different spins. Supersymmetric models with their strict fermions = bosons rule must somehow wiggle out of this, and they do so by means of auxiliary fields whose off-shell degrees of freedom disappear completely on-shell.

5.5. A supercurrent

As for any continuous symmetry, there should be a Noether current associated with the transformations (5.6) and the Lagrangian (5.4). This is a Majorana spinor-vector current, given by the formula

$$J_\mu = \frac{\partial}{\partial \dot\zeta}\left(\delta A \frac{\partial L}{\partial \partial^\mu A} + \delta B \frac{\partial L}{\partial \partial^\mu B} + \delta \psi \frac{\partial L}{\partial \partial^\mu \psi} - \kappa_\mu\right) \tag{5.8}$$

if $\delta L = \partial^\mu \kappa_\mu$. The term κ_μ is determined only up to terms of the form $\partial^\nu a_{\mu\nu}$ with $a_{\mu\nu} = -a_{\nu\mu}$. These possible "improvement terms" have been studied widely [19, 35], and one form [72] of the *supercurrent* is

$$J_\mu = \not\partial(A - \gamma_5 B)\gamma_\mu\psi + im\gamma_\mu(A - \gamma_5 B)\psi + ig\gamma_\mu(A - \gamma_5 B)^2\psi . \tag{5.9}$$

This is derived like this: for the "off-shell" Lagrangian (5.2) we get

$$\delta L = i\bar\zeta\not\partial\left[\tfrac{1}{2}\phi\,\mathsf{T}\phi - \frac{m}{2}\phi\cdot\phi - \frac{g}{3}\phi\cdot\phi\cdot\phi\right]_\psi + \tfrac{1}{4}\Box\,\delta(A^2 + B^2)$$

$$= \partial^\mu \kappa_\mu$$

with

$$-\kappa_\mu = \tfrac{1}{4}\bar\zeta(A + \gamma_5 B)[\gamma_\mu, \not\partial]\psi - \tfrac{1}{2}\bar\zeta\partial_\mu(A + \gamma_5 B)\,\psi - \tfrac{1}{2}i\bar\zeta(F + \gamma_5 G)\,\gamma_\mu\psi$$

$$+ im\bar\zeta(A + \gamma_5 B)\gamma_\mu\psi + ig\bar\zeta\gamma_\mu(A + \gamma_5 B)^2\psi .$$

The result for the "on-shell" Lagrangian (5.4) is the same, with F and G expressed in terms of A and B as given by (5.3). The other terms in J_μ are

$$\delta A \frac{\partial L}{\partial \partial^\mu A} + \delta B \frac{\partial L}{\partial \partial^\mu B} + \delta \psi \frac{\partial L}{\partial \partial^\mu \psi} = \bar\zeta \partial_\mu(A + \gamma_5 B)\psi + \tfrac{1}{2}i\bar\zeta(F + \gamma_5 G)\gamma_\mu\psi + \tfrac{1}{2}\bar\zeta\not\partial\gamma_\mu(A + \gamma_5 B)\psi .$$

Adding up, we get (5.9) for J_μ, modulo an improvement term

$$\partial^\nu a_{\mu\nu} = -\tfrac{1}{2}i\partial^\nu[\sigma_{\mu\nu}(A + \gamma_5 B)\psi] .$$

The supercurrent is conserved due to the equations of motion (5.5), as it should be. To verify this is a bit tricky and involves an identity which is discussed in the Technical Appendix, eq. (A.48).

The existence of a conserved supercurrent for a model with interaction is a highly non-trivial occurrence. In fact, we could have started with the Lagrangian (5.4), pulled the current (5.9) out of a hat, observed that it is conserved and then considered the charges

$$Q = \int d^3 x \, J_0 . \tag{5.10}$$

The transformation laws of the fields could then have been calculated as

$$\delta\phi = -i[\phi, \bar\zeta Q] ,$$

and if we had used the canonical equal-time commutation relations

$$[A, \dot{A}] = [B, \dot{B}] = i\delta; \qquad [A, A] = [B, B] = 0; \qquad \{\psi, \bar{\psi}\} = \gamma_0 \delta,$$

the transformation laws (5.6) would have emerged. The supersymmetry algebra could then have been derived from these transformation laws. This approach to supersymmetry is quite popular and widely used.

It is important that the supercurrent should actually exist for an *interacting model*. Take the free toy model

$$L_{toy} = \tfrac{1}{2}\partial^\mu A \, \partial_\mu A + \tfrac{1}{2}i\bar{\psi}\,\partial\!\!\!/\,\psi.$$

This Lagrangian has a supersymmetry

$$\delta A = \bar{\zeta}\psi; \qquad \delta\psi = -i\,\partial\!\!\!/\,A \, \zeta$$

with a current

$$J_\mu = \partial\!\!\!/\,A \, \gamma_\mu \psi$$

which is conserved due to the free equations of motion $\Box A = \partial\!\!\!/\,\psi = 0$. There is, however, no way to make this toy model interact and preserve a conserved supercurrent: in contrast to the Wess–Zumino model, the toy model is not supersymmetric in a non-trivial sense. It cannot be since it violates the "fermions = bosons rule".

6. Spontaneous supersymmetry breaking (I)

In this section, we analyse possible self-interactions of chiral multiplets and we see how it is possible to break supersymmetry spontaneously.

6.1. The superpotential

It is natural to break up the Lagrangian (5.1) of the Wess–Zumino model into a "super"-kinetic and a superpotential part:

$$L = \tfrac{1}{2}(\phi \cdot \mathsf{T}\phi)_F - [V(\phi)]_F \tag{6.1}$$

with

$$V(\phi) = \lambda\phi + \frac{m}{2}\phi \cdot \phi + \frac{g}{3}\phi \cdot \phi \cdot \phi. \tag{6.2}$$

We here include the possible $\lambda\phi$ term in the Lagrangian and will therefore see exactly why and how it can be eliminated. In this notation, the equations of motion (5.3) take the very compact form

$$\mathsf{T}\phi = V'(\phi) \tag{6.3}$$

with V' the derivative of the function V with respect to its argument.

Since all terms in $[V(\phi)]_F$ are either linear in F or linear in G or quadratic in ψ (this will be true as long as we stick to renormalisable couplings), we have a scaling equation

$$[V(\phi)]_F = (F \, \partial/\partial F + G \, \partial/\partial G + \tfrac{1}{2}\bar{\psi} \, \partial/\partial\bar{\psi})[V(\phi)]_F. \tag{6.4}$$

Let us evaluate $(\partial/\partial F)[V(\phi)]_F$ in a general way: we use a suitable adaptation of the chain rule, with $\cdot$ denoting the product of two chiral multiplets, and get

$$\frac{\partial}{\partial F} V(\phi) = \frac{\partial\phi}{\partial F} \cdot \frac{\mathrm{d}}{\mathrm{d}\phi} V(\phi) = (0,0;0;1,0) \cdot V'(\phi) = (0,0;0;[V'(\phi)]_A, -[V'(\phi)]_B).$$

This implies the first step in the following equation

$$\frac{\partial}{\partial F}[V(\phi)]_F = [V'(\phi)]_A = F, \tag{6.5a}$$

the second being the equation of motion (6.3). Similarly, we get

$$\frac{\partial}{\partial G}[V(\phi)]_F = [V'(\phi)]_B = G, \qquad \frac{\partial}{\partial\bar{\psi}}[V(\phi)]_F = [V'(\phi)]_\psi. \tag{6.5b}$$

If we now use the letters F and G, not to denote independent fields but as a shorthand for $[V'(\phi)]_A$ and $[V'(\phi)]_B$, then the superpotential becomes

$$[V(\phi)]_F = F^2 + G^2 + \tfrac{1}{2}\bar{\psi}[V'(\phi)]_\psi. \tag{6.6}$$

The "true" potential U must, of course, include the non-kinetic terms in $\tfrac{1}{2}(\phi \cdot \mathsf{T}\phi)_F$:

$$U = -\tfrac{1}{2}(F^2 + G^2) + [V(\phi)]_F = \tfrac{1}{2}(F^2 + G^2) + \tfrac{1}{2}\bar{\psi}[V'(\phi)]_\psi. \tag{6.7}$$

We see that for the vacuum, where $\langle\psi\rangle = 0$ (otherwise Lorentz invariance would be spontaneously broken), U is never negative. This is a consequence of the positivity of the energy in any supersymmetric theory, spontaneously broken or not.

6.2. Condition for spontaneous supersymmetry breaking

Supersymmetry will be spontaneously broken if and only if the lowest energy (vacuum energy) is larger than zero because

$$\langle 0|E|0\rangle \neq 0 \Leftrightarrow Q_\alpha|0\rangle \neq 0, \tag{6.8}$$

due to

$$E_{\min} \equiv \langle 0|E|0\rangle = \frac{1}{4}\sum_{\alpha=1}^{4} |Q_\alpha|0\rangle|^2. \tag{6.9}$$

Since, on the other hand,

$$E_{\min} = \langle 0|U|0 \rangle \tag{6.10}$$

we find that

supersymmetry is spontaneously broken if and only if the minimum of the potential is positive.

This can be summarised in figs. 6.1 and 6.2.

Let us analyse the Wess–Zumino model in this respect. The potential is $\langle U \rangle = \frac{1}{2}\langle F \rangle^2 + \frac{1}{2}\langle G \rangle^2$ and

$$F = \lambda + mA + g(A^2 - B^2), \qquad G = mB + 2gAB.$$

The minimum of the potential is at $\langle F \rangle = \langle G \rangle = 0$, i.e. at

$$\left.\begin{array}{l} \langle A \rangle = -\dfrac{1}{2g}\,(m \pm \sqrt{m^2 - 4g\lambda}) \\[2mm] \langle B \rangle = 0 \end{array}\right\} \quad \text{for } m^2 \ge 4g\lambda \tag{6.11a}$$
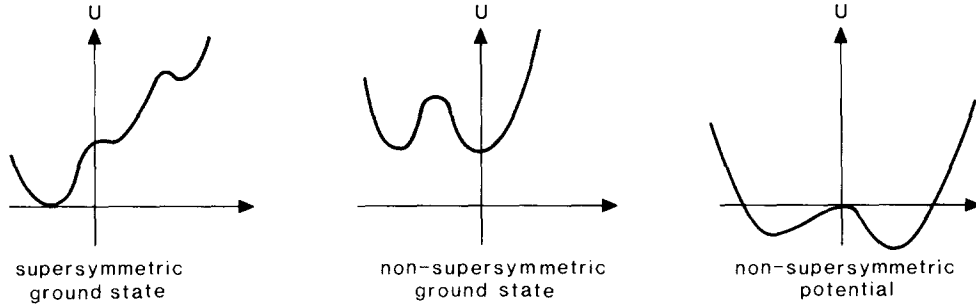
Fig. 6.1. Various potentials and their implications for supersymmetry.
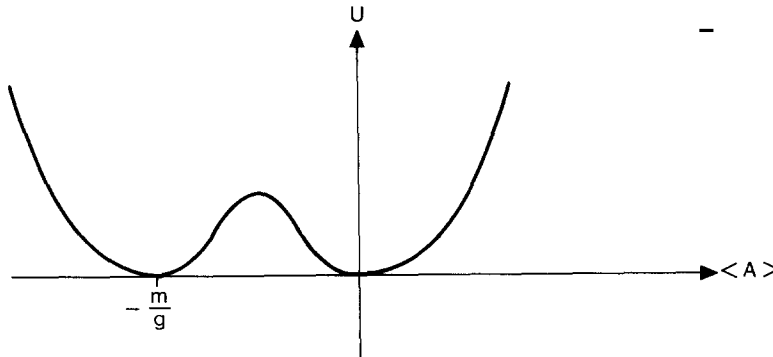
Fig. 6.2. The potential in the Wess–Zumino model.

$$\left.\begin{array}{l} \langle A\rangle = -\dfrac{m}{2g} \\[4mm] \langle B\rangle = \pm\dfrac{1}{2g}\sqrt{4g\lambda - m^2} \end{array}\right\} \quad \text{for } m^2 \leq 4g\lambda . \tag{6.11b}$$

The field redefinitions*

$$A^{\text{new}} = A - \langle A\rangle, \qquad B^{\text{new}} = B - \langle B\rangle$$

will then eliminate $\lambda\phi$ from the Lagrangian and shift one of the minima to $A = B = 0$. Figure 6.2 shows the behaviour of the potential in the Wess–Zumino model, assuming $\lambda = 0$ and $mg > 0$.

6.3. The O'Raifeartaigh model

A generalisation of the Wess–Zumino model is the following renormalisable Lagrangian for several chiral multiplets ϕ_a:

$$L = \tfrac{1}{2}(\phi_a \cdot \mathsf{T}\phi_a)_F - (V)_F \tag{6.12a}$$

with

$$V = \lambda_a\phi_a + \tfrac{1}{2}m_{ab}\phi_a \cdot \phi_b + \tfrac{1}{3}g_{abc}\phi_a \cdot \phi_b \cdot \phi_c . \tag{6.12b}$$

The λ, m and g are real and totally symmetric. The equations of motion are now

$$\mathsf{T}\phi_a = \frac{\partial}{\partial\phi_a} V = \lambda_a + m_{ab}\phi_b + g_{abc}\phi_b \cdot \phi_c \tag{6.13}$$

and the potential is

$$U = \tfrac{1}{2}F_aF_a + \tfrac{1}{2}G_aG_a + \tfrac{1}{2}\bar{\psi}_a\left[\frac{\partial}{\partial\phi_a} V\right]_\psi . \tag{6.14}$$

We shall see that the condition that there be no spontaneous breaking of supersymmetry, namely

$$\langle F_a\rangle = \langle G_a\rangle = 0 , \tag{6.15}$$

can now not always be fulfilled.

Let us consider the example given by O'Raifeartaigh [50], a special case involving three multiplets and the following assignments for the constants which define the model:

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* A further redefinition $\psi^{\text{new}} = \exp(-\tfrac{1}{4}\pi\gamma_5)\psi$ is necessary to recover manifest parity invariance if $\langle B\rangle \neq 0$.

$$\lambda_3 = \lambda \, ; \qquad \text{all other } \lambda_a = 0$$

$$m_{12} = m_{21} = m \, ; \qquad \text{all other } m_{ab} = 0 \qquad\qquad (6.16)$$

$$g_{113} = g_{131} = g_{311} = g \, ; \qquad \text{all other } g_{abc} = 0 \, .$$

The equations of motion for the six auxiliary fields are then

$$F_1 = mA_2 + 2g(A_3A_1 - B_3B_1)$$

$$F_2 = mA_1$$

$$F_3 = \lambda + g(A_1^2 - B_1^2)$$

$$G_1 = mB_2 + 2g(A_3B_1 + B_3A_1) \qquad\qquad (6.17)$$

$$G_2 = mB_1$$

$$G_3 = 2gA_1B_1$$

and we cannot set F_3, F_2 and G_2 to zero simultaneously.

Consequently, the bosonic potential

$$2U = \lambda^2 + F_1^2 + G_1^2 + (m^2 + 2g\lambda)A_1^2 + (m^2 - 2g\lambda)B_1^2 + g^2(A_1^2 + B_1^2)^2 \qquad\qquad (6.18)$$

is always positive with a possible minimal value of*

$$2U_{\text{min}} = \lambda^2 \qquad \text{for} \qquad |2g\lambda| \le m^2 \, . \qquad\qquad (6.19)$$

This value of U is achieved by the choice

$$\langle A_1 \rangle = \langle B_1 \rangle = \langle A_2 \rangle = \langle B_2 \rangle = 0 \qquad\qquad (6.20)$$

with $\langle A_3 \rangle$ and $\langle B_3 \rangle$ arbitrary, say

$$\langle A_3 \rangle = \mu/2g \, ; \qquad \langle B_3 \rangle = 0 \qquad\qquad (6.21)$$

(we set $\langle B_3 \rangle = 0$ for simplicity; this can generally be achieved by suitable field redefinitions). We plot the potential as function of $\langle A_1 \rangle$ in fig. 6.3. With the assignments (6.20–21) for the vacuum expectation values of the boson fields, we find for the fermion mass matrix

$$M_\psi = \begin{pmatrix} \mu & m & 0 \\ m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \qquad\qquad (6.22)$$

This has three different eigenvalues[†] $m_{\tilde\psi_1}$, $-m_{\tilde\psi_2}$ and $m_{\tilde\psi_3}$ which are

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* If $|2g\lambda| > m^2$, the potential will have two degenerate minima which both break supersymmetry.

[†] The tilde denotes linear combinations of the original fields.
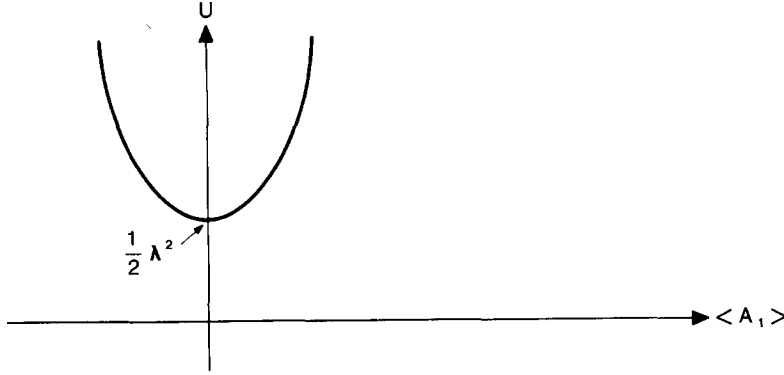
Fig. 6.3. The potential in the O'Raifeartaigh model.

$$m_{\tilde{\psi}_{1,2}} = \sqrt{\tfrac{1}{4}\mu^2 + m^2} \pm \mu/2$$

$$m_{\tilde{\psi}_3} = 0 . \tag{6.23}$$

The mass matrix for the A-fields, on the other hand, is

$$M_a^2 = \begin{pmatrix} \mu^2 + m^2 + 2g\lambda & m\mu & 0 \\ m\mu & m^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.24}$$

with eigenvalues

$$m_{A_{1,2}}^2 = \mu^2/2 + m^2 + g\lambda \pm \sqrt{\tfrac{1}{4}(\mu^2 + 2g\lambda)^2 + \mu^2 m^2}$$

$$= (\mu/2 \pm \sqrt{\tfrac{1}{4}(\mu + 2g\lambda/\mu)^2 + m^2})^2 - \lambda^2 g^2/\mu^2$$

$$= m_{\tilde{\psi}_{1,2}}^2 + g\lambda(1 \pm \mu(\mu^2 + 4m^2)^{-1/2}) + o(g^2\lambda^2) \tag{6.25}$$

$$m_{A_3}^2 = 0 .$$

The situation for the B-fields is exactly the same, except that $g\lambda$ must be replaced by $-g\lambda$ everywhere. In the interesting case of very large positive μ, i.e. for

$$\sqrt{|2g\lambda|} \leq |m| \ll \mu \tag{6.26}$$

we have a large mass scale, μ, and a small one, m^2/μ. The mass-splitting within the multiplets is of order $g\lambda/\mu$ and may well be almost as large as the small mass scale itself. In this approximation, we get

$$m_{\tilde{\psi}_1} \approx \mu; \qquad m_{A_1} \approx m_{\tilde{\psi}_1} + g\lambda/\mu, \qquad m_{B_1} \approx m_{\tilde{\psi}_1} - g\lambda/\mu$$

$$m_{\tilde{\psi}_2} \approx m^2/\mu; \qquad m_{A_2} \approx m_{\tilde{\psi}_2} + g\lambda/\mu, \qquad m_{B_2} \approx m_{\tilde{\psi}_2} - g\lambda/\mu . \tag{6.27}$$

Figure 6.4 illustrates this spectrum. In the tree approximation, the "large hierarchy limit"
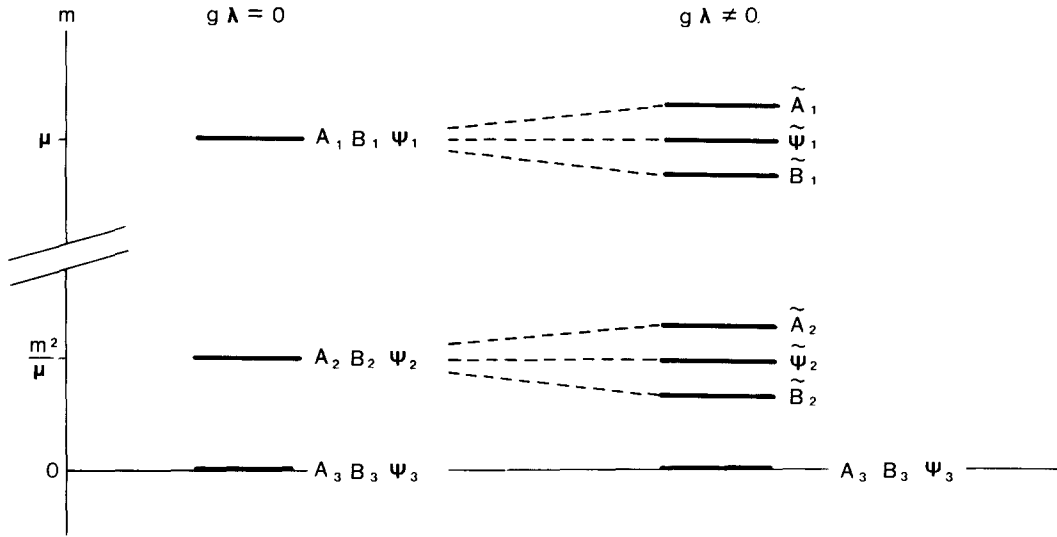
Fig. 6.4. Mass spectrum of the O'Raifeartaigh model in the "large hierarchy limit".

$$\langle A_3 \rangle = \mu/2g \gg \text{other masses}$$

is not at all unnatural. The vacuum degeneracy in A_3 which allowed us to consider this case is, however, broken by quantum effects [38, 77] and the O'Raifeartaigh model actually does not generate a hierarchy: U grows logarithmically for large A_3 and the minimum of the potential is at relatively small μ. This is so because pure spin $\frac{1}{2}$-spin 0 models are not asymptotically free.

6.4. The mass formula

For $g\lambda = 0$, i.e. in the absence of spontaneous supersymmetry breaking, the Hermitian matrices M^2_A, M^2_B and M_ψ commute with each other and can be simultaneously diagonalised. The resulting multiplets are mass-degenerate, as expected, with masses given by eq. (6.23). Accordingly, we find

$$M^2_A = M^2_B = (M_\psi)^2 .$$

For $g\lambda \neq 0$, i.e. in the presence of supersymmetry breaking, only a weaker condition

$$M^2_A + M^2_B = 2(M_\psi)^2 \tag{6.28}$$

still holds true and no two of the Hermitian matrices M^2_A, M^2_B and $(M_\psi)^2$ can be simultaneously diagonalised. The mass eigenstates are therefore different mixtures of multiplets ϕ_1 and ϕ_2 for A, B and ψ and in general no two of the six particles $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{\psi}_1$ and $\tilde{\psi}_2$ have the same mass. From the trace of eq. (6.28) for the mass matrices, however, we find the important relationship

$$\sum_{\substack{\text{all boson} \\ \text{states}}} m_i^2 = \sum_{\substack{\text{all fermion} \\ \text{states}}} m_i^2 \tag{6.29a}$$

or, since the number of states for a charge self-conjugate spin-s particle is $(2s + 1)$,

$$\sum_i (-1)^{2s}(2s + 1)m_i^2 = 0 \, ,$$ (6.29b)

where the index i runs over all charge self-conjugate particles in the model. In the case of unbroken supersymmetry, this is trivially fulfilled because for each multiplet in the model the masses are degenerate and because of the "fermions = bosons" rule, which can be written as

$$\sum_i (-1)^{2s}(2s + 1) = 0 \, .$$ (6.30)

7. $N = 1$ superspace

A compact and very useful technique for working out representations of the supersymmetry algebra on fields was invented by A. Salam and J. Strathdee [55]: the superfield in superspace. It is particularly useful for $N = 1$ theories; their superfield structure is completely known. The technique is less well developed for extended supersymmetry, although it appears that important progress has been made recently for $N = 2$ [23] beyond earlier superspace work [29, 46, 36].

$N = 1$ superspace has coordinates x^μ, θ^α and $\bar{\theta}^{\dot\alpha}$ and serves to represent the algebra of $N = 1$ Poincaré supersymmetry in very much the same way as the Poincaré algebra is represented on ordinary space–time.

7.1. Minkowski space (a toy example)

In order to get a feeling for what will happen, let us first consider ordinary quantum fields $\phi(x)$ which depend only on the four coordinates x^μ of Minkowski space. Translations of these coordinates are generated by the operators P_μ in the usual way, eq. (3.42), and we can consider $\phi(x)$ to have been translated from $x^\mu = 0$:

$$\phi(x) = e^{ix \cdot P} \phi(0) e^{-ix \cdot P} \, .$$ (7.1)

The transformation law (7.1) is compatible with the multiplication law

$$e^{iy \cdot P} e^{ix \cdot P} = e^{i(x+y) \cdot P} \, ,$$ (7.2)

which holds because the operators P_μ commute with each other.

The expression on the right-hand side of eq. (7.1) can be expanded by use of the formula

$$e^{-B} A \, e^{B} = \sum_{n=0}^{\infty} \frac{1}{n!}[A, B]_{(n)}$$ (7.3a)

with

$$[A, B]_{(0)} = A; \qquad [A, B]_{(n+1)} = [[A, B]_{(n)}, B] \, ,$$ (7.3b)

and then says that $\phi(x)$ is completely determined by the properties at any given point x_0 (here $x_0 = 0$), namely by the multiple commutators of $\phi(x_0)$ with P_μ to arbitrary order.

For a Lorentz rotation we have

$$\exp\left(\frac{i}{2}\lambda \cdot M\right) \phi(x) \exp\left(-\frac{i}{2}\lambda \cdot M\right) = \exp(ix' \cdot P) \exp\left(\frac{i}{2}\lambda \cdot M\right) \phi(0) \exp\left(-\frac{i}{2}\lambda \cdot M\right) \exp(-ix' \cdot P),$$

(7.4a)

where $\lambda \cdot M \equiv \lambda^{\mu\nu} M_{\mu\nu}$, and x' is given implicitly by

$$\exp\left(\frac{i}{2}\lambda \cdot M\right) \exp(ix \cdot P) = \exp(ix' \cdot P) \exp\left(\frac{i}{2}\lambda \cdot M\right).$$

(7.4b)

For infinitesimal λ, we can easily calculate x',

$$\left(1 + \frac{i}{2}\lambda \cdot M\right) e^{ix \cdot P} = e^{ix \cdot P} e^{-ix \cdot P}\left(1 + \frac{i}{2}\lambda \cdot M\right) e^{ix \cdot P}$$

$$= e^{ix \cdot P}\left(1 + \frac{i}{2}\lambda \cdot M + \left[\frac{i}{2}\lambda \cdot M, ix \cdot P\right]\right)$$

$$= \exp\{i(x^\mu + x^\nu \lambda_\nu{}^\mu)P_\mu\} \left(1 + \frac{i}{2}\lambda \cdot M\right) + o(\lambda^2),$$

making use of eqs. (7.3) and the algebra of the Poincaré group. For finite $\lambda_\mu{}^\nu$, we get the expected transformation law for the coordinates under Lorentz transformations:

$$x'^\mu = x^\nu (e^\lambda)_\nu{}^\mu = (e^{-\lambda})^\mu{}_\nu x^\nu.$$

(7.5)

Finally, the action of the Lorentz transformation on $\phi(0)$ can at most be some linear transformation which acts on as yet unwritten indices which $\phi(0)$, and hence $\phi(x)$, may have:

$$\exp\left(\frac{i}{2}\lambda \cdot M\right) \phi(0) \exp\left(-\frac{i}{2}\lambda \cdot M\right) = \exp\left(-\frac{i}{2}\lambda \cdot \Sigma\right) \phi(0).$$

(7.6)

Here the $\Sigma_{\mu\nu}$ are some matrix representations of the algebra of the $M_{\mu\nu}$. The minus sign on the right-hand side is crucial: a further Lorentz transformation does not see the numerical matrices Σ and will act "inside", directly on $\phi(0)$ – this reverses the factors on the right-hand side against those on the left and we need the minus sign to retain the group multiplication law. All in all, we get

$$\exp\left(\frac{i}{2}\lambda \cdot M\right) \phi(x) \exp\left(-\frac{i}{2}\lambda \cdot M\right) = \exp\left(-\frac{i}{2}\lambda \cdot \Sigma\right) \phi(e^{-\lambda}x).$$

(7.7)

The differential version of eq. (7.7) is

$$[\phi(x), M_{\mu\nu}] = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \phi(x) + \Sigma_{\mu\nu} \phi(x).$$

(7.8)

Before going on to repeat the same procedure in superspace for the superalgebra (2.38), let us try to understand in a more formal manner what happened here.

7.2. Coset spaces

Let g be an arbitrary element of a group G (in our particular example of the previous subsection this was the Poincaré group) which contains a subgroup H (here the Lorentz group). We now define equivalence classes of elements of G: two elements g and g' are considered equivalent if they can be connected by a right multiplication with an element $h \in$ H:

$$g' = g \circ h \quad \text{or} \quad g^{-1} \circ g' \in H. \tag{7.9}$$

This equivalence class is called the (*left*) *coset* of g with respect to H. The set of all cosets is a *manifold*, denoted by G/H.

A set of group elements $L(x)$, labelled by as many parameters as necessary, parametrizes the manifold if each coset contains exactly one of the L's. Once we have chosen a parametrization $L(x)$, each group element g can be uniquely decomposed into a product

$$g = L(x) \circ h \tag{7.10}$$

where L is the representative member of the coset to which g belongs and h connects L to g within the coset. A product of g with an arbitrary group element, and in particular with some $L(x)$, will thus define another L and an h according to

$$g \circ L(x) = L(x') \circ h. \tag{7.11}$$

Here x' and h are in general functions of both x and g:

$$x' = x'(x, g), \quad h = h(x, g). \tag{7.12}$$

The Minkowski space of the previous subsection was the coset manifold Poincaré/Lorentz. We had parametrized it by

$$L(x) = e^{ix \cdot P} \tag{7.13}$$

and determined x' and h explicitly as

$$\left. \begin{array}{l} x' = x + y \\ h = 1 \end{array} \right\} \quad \text{for } g = L(y), \text{ see eq. (7.2)}$$

$$\left. \begin{array}{l} x' = e^{-\lambda}x \\ h = g \end{array} \right\} \quad \text{for } g = \exp(\tfrac{1}{2}i\lambda \cdot M) \in H, \text{ see eqs. (7.4-5)}. \tag{7.14}$$

That case was particularly simple because $[P, P] = 0$ and $[P, M] = P$. The quantum fields were written as

$$\phi(x) = L(x)\,\phi(0)\,L^{-1}(x) \tag{7.15}$$

and the action of a group element on them was completely determined,

$$g \, \phi(x) \, g^{-1} = L(x') \, h \, \phi(0) \, h^{-1} \, L^{-1}(x') \,, \tag{7.16}$$

once we knew how h acted on $\phi(0)$:

$$h \, \phi(0) \, h^{-1} = \exp(-\tfrac{1}{2} i \lambda \cdot \Sigma) \, \phi(0) \,. \tag{7.17}$$

The case of the coset Poincaré/Lorentz is not the most general example of a coset space: due to the fact that the translations form an invariant Abelian subgroup, the "little group" element h in (7.11) is actually independent of x, see (7.14). This will not be so for all coset spaces (compare the representation of the conformal group on fields over Minkowski space in section 16).

7.3. Need for anticommuting parameters

We now turn our attention to the supersymmetry algebra. Immediately, we stumble over a problem: it is not the Lie algebra of a group since it involves anticommutators. We must, however, be able to exponentiate the algebra into a group in such a way that the product of two group elements is again a group element, say

$$e^{i\theta Q} \, e^{i\bar{Q}\bar{\theta}} = e^{\text{(something)}} \,.$$

In the general case of non-commuting objects this required behaviour is described by the Baker–Campbell–Haussdorff formula,

$$e^A \, e^B = \exp\!\left[\sum_{n=1}^{\infty} \frac{1}{n!} C_n(A, B) \right], \tag{7.18a}$$

which can be derived using nothing else but the associativity of multiplication. The coefficients $C_n(A, B)$ are multi-commutators of A and B. The first five of these are

$$
\begin{aligned}
&C_1 = A + B \\
&C_2 = [A, B] \\
&C_3 = \tfrac{1}{2}[[A, B], B] + \tfrac{1}{2}[A, [A, B]] \\
&C_4 = [[A, [A, B]], B] \\
&C_5 = -\tfrac{1}{6}[A, [A, [A, [A, B]]]] - \tfrac{1}{6}[[[[A, B], B], B], B] - \tfrac{1}{3}[[A, [A, [A, B]]], B] \\
&\qquad -\tfrac{1}{3}[A, [[[A, B], B], B]] + [A, [[A, [A, B]], B]] + [[A, [[A, B], B]], B] \,.
\end{aligned}
\tag{7.18b}
$$

Here and in all other C_n only commutators appear, no anticommutators, and we seem to be in trouble. It is, however, a straightforward exercise to show that commutators like $[\theta Q, \bar{Q}\bar{\theta}]$ can be reduced to expressions which only involve the anticommutators of the Q's if we assume the parameters θ^α and $\bar{\theta}^{\dot\alpha}$ to be anticommuting spinorial quantities, just as the ζ's were in section 5.

Strictly speaking, such anticommuting parameters can be avoided (at some cost) in the context of rigid (flat-space) supersymmetry, but not in supergravity where they appear in transformation laws like $\delta\psi_\mu = \nabla_\mu\zeta$, and they certainly cannot be avoided in any superspace treatment of the subject.

7.4. Superspace

With these parameters, the supersymmetry algebra can be integrated to a group G, the *super-Poincaré group*, with typical group elements

$$g = \exp(\mathrm{i}x \cdot P + \mathrm{i}\theta Q + \mathrm{i}\bar{Q}\bar{\theta} + \tfrac{1}{2}\mathrm{i}\lambda \cdot M).$$ (7.19)

Superspace is the coset space super-Poincaré/Lorentz. There are infinitely many different ways to parametrize this manifold, the one most commonly used is "real" or "symmetric" superspace, parametrized by

$$L(x, \theta, \bar{\theta}) = \exp(\mathrm{i}x \cdot P + \mathrm{i}\theta^\alpha Q_\alpha + \mathrm{i}\bar{Q}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}).$$ (7.20)

The space clearly has eight coordinates, four bosonic ones x^μ and four fermionic ones θ^α if we assume $\bar{\theta}^{\dot{\alpha}} = (\theta^\alpha)^*$.

The multiplication laws (7.11) can be evaluated to give, for the multiplication with a second L,

$$L(y, \zeta, \bar{\zeta}) L(x, \theta, \bar{\theta}) = L(x', \theta', \bar{\theta}')$$ (7.21a)

with transformed coordinates

$$\begin{aligned} x'^\mu &= x^\mu + y^\mu + \mathrm{i}\zeta\sigma^\mu\bar{\theta} - \mathrm{i}\theta\sigma^\mu\bar{\zeta} \\ \theta' &= \theta + \zeta; \qquad \bar{\theta}' = \bar{\theta} + \bar{\zeta}. \end{aligned}$$ (7.21b)

Multiplication with an element of the Lorentz group gives

$$\exp(\tfrac{1}{2}\mathrm{i}\lambda \cdot M) L(x, \theta, \bar{\theta}) = L(x', \theta', \bar{\theta}') \exp(\tfrac{1}{2}\mathrm{i}\lambda \cdot M)$$ (7.22a)

with transformed coordinates

$$\begin{aligned} x'^\mu &= (\mathrm{e}^{-\lambda})^\mu{}_\nu x^\nu \\ \theta' &= \theta \exp(-\tfrac{1}{4}\mathrm{i}\lambda^{\mu\nu}\sigma_{\mu\nu}); \qquad \bar{\theta}' = \exp(\tfrac{1}{4}\mathrm{i}\lambda^{\mu\nu}\bar{\sigma}_{\mu\nu})\bar{\theta}. \end{aligned}$$ (7.22b)

The evaluation of eq. (7.21a) involves use of eq. (7.18) and the additional θ-dependent terms in x' are a consequence of the non-vanishing anticommutator $\{Q, \bar{Q}\}$. They are an essential feature of any realization of the supersymmetry algebra on a manifold.

7.5. Representations on superfields

A superfield is defined in complete analogy with eq. (7.15) as

$$\phi(x, \theta, \bar{\theta}) = L(x, \theta, \bar{\theta}) \, \phi(0, 0, 0) \, L^{-1}(x, \theta, \bar{\theta}) \,, \tag{7.23}$$

and for any group element the action on ϕ is given by the coordinate transformations (7.21–22) in conjunction with

$$\exp(\tfrac{1}{2}i\lambda \cdot M) \, \phi(0, 0, 0) \exp(-\tfrac{1}{2}i\lambda \cdot M) = \exp(-\tfrac{1}{2}i\lambda \cdot \Sigma) \, \phi(0, 0, 0) \,. \tag{7.24}$$

In its infinitesimal form, we write the group action on a superfield $\phi = \phi(x, \theta, \bar{\theta})$ as

$$\begin{aligned}
\delta_\zeta \phi &= -i[\phi, \zeta Q] = -i\zeta \, r(Q) \, \phi \\
\delta_{\bar{\zeta}} \phi &= -i[\phi, \bar{Q}\bar{\zeta}] = i\bar{\zeta}^{\dot\alpha} \, r(\bar{Q}_{\dot\alpha}) \, \phi \\
\delta_y \phi &= -i[\phi, y \cdot P] = -iy \cdot r(P) \, \phi \\
\delta_\lambda \phi &= -\tfrac{1}{2}i[\phi, \lambda \cdot M] = -\tfrac{1}{2}i\lambda \cdot r(M) \, \phi
\end{aligned} \tag{7.25}$$

with the differential operator representation r of the algebra given by

$$\begin{aligned}
r(Q_\alpha) &= i \, \partial/\partial\theta^\alpha - (\sigma^\mu \bar{\theta})_\alpha \partial_\mu \\
r(\bar{Q}_{\dot\alpha}) &= -i \, \partial/\partial\bar{\theta}^{\dot\alpha} + (\theta\sigma^\mu)_{\dot\alpha} \partial_\mu \\
r(P_\mu) &= i \, \partial_\mu \\
r(M_{\mu\nu}) &= i \, (x_\mu \partial_\nu - x_\nu \partial_\mu) + \tfrac{1}{2}\theta\sigma_{\mu\nu}\partial/\partial\theta - \tfrac{1}{2}(\bar{\theta}\bar{\sigma}_{\mu\nu})^{\dot\alpha} \, \partial/\partial\bar{\theta}^{\dot\alpha} + \Sigma_{\mu\nu} \,.
\end{aligned} \tag{7.26}$$

The optional chiral transformations which are generated by R, cf. eq. (2.39), act as

$$\delta_\alpha \phi = -i[\phi, \alpha R] = -i\alpha \, r(R) \, \phi \tag{7.27}$$

with

$$r(R) = \theta^\alpha \partial/\partial\theta^\alpha - \bar{\theta}^{\dot\alpha} \, \partial/\partial\bar{\theta}^{\dot\alpha} + n \,. \tag{7.28}$$

The number n is the *chiral weight* of the superfield and must be real in order to render the R-transformations compact.

These operators represent the algebra (2.38) for $N = 1$. Great care has to be taken to get the signs right in these equations – they are rather dependent on the conventions discussed in the appendix. If ϕ does not have overall spinor indices, we can write the first two equations of (7.25) as

$$[\phi, Q] = r(Q) \, \phi; \qquad [\phi, \bar{Q}] = r(\bar{Q}) \, \phi \,.$$

The differentiations $\partial/\partial\theta$ and $\partial/\partial\bar{\theta}$ are defined by

$$\frac{\partial}{\partial\theta^\alpha} \, \theta^\beta = \delta^\beta{}_\alpha; \qquad \frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} \, \bar{\theta}^{\dot\beta} = \delta^{\dot\beta}{}_{\dot\alpha} \tag{7.29}$$

and because of the difficult covariance properties of the δ-symbol, cf. eq. (A.24), we should avoid using them with any other index positions. In order for these definitions to make sense, we must always anticommute θ or $\bar{\theta}$ to immediately behind the differentiation operator.

7.6. Component fields (general case)

The Baker–Campbell–Haussdorff formula (7.3) and the definition (7.23) mean that a superfield is actually *defined* as a Taylor-expansion in θ and $\bar{\theta}$ with coefficients which are themselves local fields over Minkowski space. Since the third powers of θQ and $Q\bar{\theta}$ are already zero, due to the vanishing of the square of each component of θ^α and $\bar{\theta}^{\dot\alpha}$, this expansion will break off rather soon, and the following is already the most general superfield (I follow standard conventions and use the letter V to denote it):

$$V(x, \theta, \bar{\theta}) = C - i\theta\chi + i\bar{\chi}'\bar{\theta} - \tfrac{1}{2}i\theta^2(M - iN) + \tfrac{1}{2}i\bar{\theta}^2(M + iN) - \theta\sigma^\mu\bar{\theta}A_\mu$$

$$+ i\bar{\theta}^2\theta(\lambda - \tfrac{1}{2}i\partial\bar{\chi}') - i\theta^2\bar{\theta}(\bar{\lambda}' - \tfrac{1}{2}i\bar{\partial}\chi) - \tfrac{1}{2}\theta^2\bar{\theta}^2(D + \tfrac{1}{2}\Box C) . \tag{7.30}$$

Assuming that it has no over-all Lorentz indices, this superfield contains as Taylor coefficients four complex scalar fields $C(x)$, $M(x)$, $N(x)$ and $D(x)$, one complex vector $A_\mu(x)$, two spinors $\chi(x)$ and $\lambda(x)$ in the $(\tfrac{1}{2}, 0)$ representation and two unrelated spinors $\bar{\chi}'(x)$ and $\bar{\lambda}'(x)$ in the $(0, \tfrac{1}{2})$ representation of the Lorentz group, altogether 16 (bosonic) + 16 (fermionic) field components.

Let us now consider *complex conjugation*. If we define

$$(\theta^\alpha)^* = \bar{\theta}^{\dot\alpha} \tag{7.31}$$

and impose as a rule that fermionic quantities should reverse their order under complex conjugation, like operators do under Hermitian conjugation, then we have

$$[L(x, \theta, \bar{\theta})]^\dagger = L^{-1}(x, \theta, \bar{\theta}) , \tag{7.32}$$

and a reality condition on a superfield

$$[V(x, \theta, \bar{\theta})]^\dagger = V(x, \theta, \bar{\theta}) \tag{7.33a}$$

is covariant since $V^\dagger$ is a superfield just as V was. We can impose such a condition on the superfield (7.30) and find that then

$$C = C^\dagger; \quad M = M^\dagger; \quad N = N^\dagger; \quad D = D^\dagger$$

$$\bar{\lambda}' = \lambda^\dagger; \quad \bar{\chi}' = \chi^\dagger , \tag{7.33b}$$

so that the total number of component fields is now only $8 + 8$. This is called the *real (general) superfield*. In order to maintain this reality condition, the supersymmetry transformations must, of course, be restricted by

$$(\zeta^\alpha)^\dagger = \bar{\zeta}^{\dot\alpha} \tag{7.34}$$

and the chiral weight of a real V must be zero.

The transformation laws under supersymmetry transformations for the components of V are calculated by comparing coefficients in the expansion $\delta V = \delta C - i\theta\,\delta\chi + \cdots$ with the $\delta_\zeta V + \delta_{\bar\zeta} V$, which is given by eqs. (7.25–26) when acting on (7.30). The result of this calculation, expressed in four-component notation, will be the transformation laws (4.7) of the real general multiplet, in other words,

the component fields of a general superfield form a general multiplet.

As we know, the general real multiplet is not an irreducible representation. Here we encounter a major problem with the superspace approach to supersymmetry: superfields do in general not correspond to irreducible representations. The latter must be obtained by imposing constraints on the superfields.

7.7. Covariant spinor derivatives

Immediately after the concept of superspace and superfields was developed, it was realized [17] that something else was needed to make contact with the irreducible representations of supersymmetry. In general, superfields contain more components than acceptable for irreducible representations, and the problem arose – and was solved in ref. [17] – of how to impose supersymmetric conditions on superfields to reduce the number of degrees of freedom which they describe.

A "supersymmetric condition" would be, for example,

$$\partial_\mu \phi(x, \theta, \bar\theta) = 0 \tag{7.35}$$

– not a very interesting one since it just leaves us a single constant to describe the entire superfield (there could be a multiplet of constants, but then Q would be represented non-trivially while P_μ was represented trivially: Q would not be an operator on a positive definite Hilbert space). More interesting conditions involve covariant spinor derivatives which can be introduced in a rather elegant way.

The associativity of group multiplication

$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3) \tag{7.36}$$

can be interpreted to mean that the "right action" of the group element g_3 on g_2 and the "left action" of g_1 on g_2 commute with each other – it does not matter in which order we perform the operations. Correspondingly, the left action of one L on another,

$$L(y, \zeta, \bar\zeta)\,L(x, \theta, \bar\theta) \approx [1 - iy^\mu\,r(P_\mu) - i\zeta\,r(Q) + i\bar\zeta^{\dot\alpha}\,r(\bar Q_{\dot\alpha})]\,L(x, \theta, \bar\theta), \tag{7.37}$$

which we interpreted as a transformation of the superfield, commutes with the right action, which we can write as:

$$L(x, \theta, \bar\theta)\,L(y, \zeta, \bar\zeta) \approx [1 + y^\mu D_\mu + \zeta^\alpha D_\alpha - \bar\zeta^{\dot\alpha}(\bar D_{\dot\alpha})]\,L(x, \theta, \bar\theta). \tag{7.38}$$

We can explicitly calculate the D's and find

$$D_\mu = \partial_\mu$$

$$D_\alpha = \partial/\partial\theta^\alpha - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu \tag{7.39}$$

$$\bar D_{\dot\alpha} = -\partial/\partial\bar\theta^{\dot\alpha} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \,.$$

The calculation is particularly simple since we need only reverse the roles of parameters and coordinates in eqs. (7.21).

Whereas the left action induces a mapping of the manifold G/H on itself which is a *realization* of the group, the right action induces an *anti-realization* (the order is inverted). Without any calculation, the algebra of the D's can be given: the associative law (7.36) implies

$$\{D, r(Q)\} = \{\bar D, r(Q)\} = 0$$
$$\{D, r(\bar Q)\} = \{\bar D, r(\bar Q)\} = 0\,, \tag{7.40}$$

and the commutators of the D's with each other

$$\{D, D\} = \{\bar D, \bar D\} = 0$$
$$\{D_\alpha, \bar D_\beta\} = 2i\sigma^\mu{}_{\alpha\beta}\partial_\mu \tag{7.41}$$
$$[D, \partial_\mu] = [\bar D, \partial_\mu] = [\partial_\mu, \partial_\nu] = 0$$

resemble the fact that the right action induces an anti-realization on superspace. Having defined the D's off by a factor of i from $r(Q)$, this algebra is formally the same as that for the $r(Q)$.

The conventions for D and $\bar D$ were chosen such that

$$(D_\alpha\phi)^\dagger = \bar D_{\dot\alpha}\bar\phi \quad\text{if}\quad (\phi)^\dagger = \bar\phi \quad\text{and } \phi \text{ bosonic}\,. \tag{7.42}$$

There is a difference in the reality conditions between $r(Q)$ and $r(\bar Q)$ on the one hand and D and $\bar D$ on the other: the former are "Hermitian adjoint" to each other in the same sense in which $i\partial_\mu$ is Hermitian, in an inner-product space with integration as inner product. The latter are "complex conjugate" to each other in the same sense in which ∂_μ is real, the gradient of a real field being real. Thus, although the D's seem to have the same algebra* as the $r(Q)$'s, they do *not* correspond to another supersymmetry: as representations of symmetry operators, we would have to take iD and $i\bar D$. These have the opposite algebra and the energy P_0 would not have a definite sign if we interpreted them as representations of operators in a Hilbert space of physical states.

The eqs. (7.40) have as a consequence that if $\phi(x, \theta, \bar\theta)$ is a superfield, i.e. if it is of the form (7.23) and transforms according to (7.25–26), then so are $D\phi$, $\bar D\phi$ and $\partial_\mu\phi$. The only change is in the representation matrix $\Sigma_{\mu\nu}$ of the Lorentz generators since the derivatives carry Lorentz indices, and in the chiral weight n which is raised by one for $D\phi$ and lowered by one for $\bar D\phi$. These derivatives are therefore covariant under super-Poincaré transformations of the superspace coordinates.

7.8. Chiral superfields and chiral parametrization

Like all covariant derivatives, D and $\bar D$ can be used to impose *covariant conditions* on superfields.

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* Other authors may use different conventions.

The simplest, most widely used and most important such conditions are those for a chiral superfield ϕ with

$$\bar{D}_\alpha \phi = 0 \tag{7.43a}$$

and an anti-chiral superfield $\bar\phi$ with

$$D_\alpha \bar\phi = 0 . \tag{7.43b}$$

Note that if ϕ is chiral then $(\phi)^\dagger$ is anti-chiral, due to eq. (7.42), and that a superfield cannot be both chiral and anti-chiral except if it is a constant. This follows from the algebra (7.41). Also, a superfield cannot be both chiral (or anti-chiral) and real. The conditions (7.43) are first-order differential equations and can easily be solved:

$$\phi(x, \theta, \bar\theta) = \exp(-\mathrm{i}\theta\sigma\bar\theta)\, \phi(x, \theta), \qquad \bar\phi(x, \theta, \bar\theta) = \exp(\mathrm{i}\theta\sigma\bar\theta)\, \bar\phi(x, \bar\theta) , \tag{7.44}$$

with $\phi(x, \theta)$ and $\bar\phi(x, \bar\theta)$ not depending on $\bar\theta$ and θ, respectively. Their Taylor expansion in terms of ordinary fields is particularly simple,

$$\phi(x, \theta) = A + 2\theta\psi - \theta^2 F, \qquad \bar\phi(x, \bar\theta) = A^\dagger + 2\bar\psi\bar\theta - \bar\theta^2 F^\dagger , \tag{7.45}$$

and gives rise to irreducible multiplets which turn out to be exactly the chiral and anti-chiral multiplets (3.50) and (3.53).

If we only ever had to deal with chiral superfields, it would have been more convenient to parametrize superspace differently from (7.20). The choices [17]

$$L_{(1)}(x, \theta, \bar\theta) = \mathrm{e}^{\mathrm{i}x\cdot P}\, \mathrm{e}^{\mathrm{i}\theta Q}\, \mathrm{e}^{\mathrm{i}\bar Q\bar\theta}$$

$$L_{(2)}(x, \theta, \bar\theta) = \mathrm{e}^{\mathrm{i}x\cdot P}\, \mathrm{e}^{\mathrm{i}\bar Q\bar\theta}\, \mathrm{e}^{\mathrm{i}\theta Q} \tag{7.46}$$

are related to our previous parametrization by the shifts

$$L_{(1)}(x^\mu - \mathrm{i}\theta\sigma^\mu\bar\theta, \theta, \bar\theta) = L(x, \theta, \bar\theta) = L_{(2)}(x^\mu + \mathrm{i}\theta\sigma^\mu\bar\theta, \theta, \bar\theta) \tag{7.47}$$

and, correspondingly, the representations $r_{(i)}$ and the covariant derivatives $D_{(i)}$ take a different, shifted form:

$$r_{(1)}(Q) = \mathrm{e}^{\mathrm{i}\theta\sigma\bar\theta}\, r(Q)\, \mathrm{e}^{-\mathrm{i}\theta\sigma\bar\theta} = \mathrm{i}\partial/\partial\theta$$

$$r_{(1)}(\bar Q) = \mathrm{e}^{\mathrm{i}\theta\sigma\bar\theta}\, r(\bar Q)\, \mathrm{e}^{-\mathrm{i}\theta\sigma\bar\theta} = -\mathrm{i}\partial/\partial\bar\theta + 2\theta\sigma$$

$$D_{(1)} = \mathrm{e}^{\mathrm{i}\theta\sigma\bar\theta}\, D\, \mathrm{e}^{-\mathrm{i}\theta\sigma\bar\theta} = \partial/\partial\theta - 2\mathrm{i}\sigma\bar\theta \tag{7.48}$$

$$\bar D_{(1)} = \mathrm{e}^{\mathrm{i}\theta\sigma\bar\theta}\, \bar D\, \mathrm{e}^{-\mathrm{i}\theta\sigma\bar\theta} = -\partial/\partial\bar\theta ,$$

and similarly for $i = 2$. We see that a chiral superfield is particularly simple in the 1-parametrization where it does not depend on $\bar\theta$, whereas an anti-chiral superfield is simple in the 2-parametrization

where it does not depend on θ. The exponentials in (7.44) can be seen as the shifts from these parametrizations to the real (or "symmetric") one. Since the "chiral" parametrizations do not easily allow complex conjugation, and thus reality conditions, one usually rather tolerates the shifts in (7.44) than the strange asymmetry of (7.48) under $\theta \leftrightarrow \bar{\theta}$. We shall mostly ignore chiral parametrizations henceforth.

7.9. Submultiplets as superfields

To conclude this initial introduction to superspace, let us rewrite the results of subsection 4.3 in terms of superfields.

The curl multiplet dV corresponds to a superfield

$$W_\alpha = \frac{i}{4} \bar{D}^2 D_\alpha V. \tag{7.49}$$

This can be confirmed by establishing that the lowest, $\theta - \bar{\theta}$-independent component of W_α is the field $\lambda_\alpha(x)$, which is the lowest-dimensional member of the multiplet dV. We find that indeed

$$\frac{i}{4} \bar{D}^2 D_\alpha V \Big]_{\theta = \bar{\theta} = 0} = \frac{i}{4} \varepsilon^{\dot{\beta}\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}} (\bar{\theta}^2\text{-component of } D_\alpha V) = \lambda_\alpha$$

because

$$(\bar{\theta}^2\text{-component of } D_\alpha V) = \frac{\partial}{\partial \theta^\alpha} \left[i\bar{\theta}^2 \theta \left(\lambda - \frac{i}{2} \partial\!\!\!/\bar{x} \right) \right] - i(\partial\!\!\!/\bar{\theta})_\alpha \, (i\bar{\chi}\bar{\theta})$$

$$= i\bar{\theta}^2 \left(\lambda - \frac{i}{2} \partial\!\!\!/\bar{\chi} \right)_\alpha - \tfrac{1}{2}\bar{\theta}^2 (\partial\!\!\!/\bar{\chi})_\alpha = i\bar{\theta}^2 \lambda_\alpha \, .$$

To conclude from equality of the lowest component to the equality of an entire superfield is a very common technique which is justified by the definition (7.23) of a superfield. This definition implies that if a superfield vanishes at the superspace origin, $\phi(0, 0, 0) = 0$ in the operator sense, then it is identically zero.

A $\bar{D}$ applied to W_α would result in an expression involving $\bar{D}^3$ which vanishes due to $\{\bar{D}, \bar{D}\} = 0$ and the fact that there are only two different $\bar{D}_{\dot{\alpha}}$. Therefore W_α is a chiral superfield

$$\bar{D}_{\dot{\alpha}} W_\alpha = 0 \, . \tag{7.50}$$

Since W_α is not of the form for the most general chiral superfield with a spinor index (that would be $\bar{D}^2 V_\alpha$), it must be restricted by a further condition. This is

$$D^\alpha W_\alpha + \bar{D}_{\dot{\alpha}} W^{\dot{\alpha}} = 0 \tag{7.51}$$

which follows from the identity

$$D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} \, . \tag{7.52}$$

It is this condition which is responsible, among other things, for the $F_{\mu\nu}$ in W_α being a curl, i.e. for eq. (4.11). The full component expansion of W_α is

$$W = \exp(-i\theta\sigma\bar{\theta})[\lambda - \tfrac{1}{2}i\sigma^{\mu\nu}\theta F_{\mu\nu} + i\theta D - i\theta^2\sigma\bar{\lambda}] \,. \tag{7.53}$$

The constrained multiplet with $dV = 0$ corresponds to a superfield

$$V = \tfrac{1}{2}i(\bar{\phi} - \phi) \quad \text{with} \quad \bar{D}_{\dot{\alpha}}\phi = 0 \,. \tag{7.54}$$

This is the most general solution of $W_\alpha = 0$.

In a similar fashion, it can be shown that the chiral multiplet ∂V corresponds to a chiral superfield X which can be projected out of V:

$$X = \tfrac{1}{2}i\bar{D}^2 V \,. \tag{7.55}$$

The linear multiplet with $\partial L = 0$ has as its superfield equivalent a real L for which

$$\bar{D}^2 L = D^2 L = 0 \,. \tag{7.56}$$

The most general solution of this is

$$L = i(D^\alpha \Lambda_\alpha + \bar{D}_{\dot{\alpha}} \Lambda^{\dot{\alpha}}) \quad \text{with} \quad \bar{D}_{\dot{\alpha}}\Lambda_\alpha = 0 \,. \tag{7.57}$$

The *chiral weight* of W_α is -1, that of X is -2, that of ϕ must be 0 and that of Λ_α must be -1.

7.10. Superfield tensor calculus

In the superfield formulation, the crucial property which makes tensor calculus possible, is that the *product of two superfields* is again a superfield:

$$\phi_1(x, \theta, \bar{\theta}) \, \phi_2(x, \theta, \bar{\theta}) = \phi_3(x, \theta, \bar{\theta}) \,. \tag{7.58}$$

This can be seen as a consequence of the definition (7.23) or, alternatively, of the fact that the representation (7.26) of the super-algebra is in terms of first-order differential operators.

The superfield expressions for the four products (4.18–25) are

$$
\begin{aligned}
\phi_3 &= \phi_1 \cdot \phi_2 &&\Leftrightarrow& \phi_3 &= \phi_1\phi_2 \\
V &= \phi_1 \times \phi_2 &&\Leftrightarrow& V &= \tfrac{1}{2}(\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2) \\
V &= \phi_1 \wedge \phi_2 &&\Leftrightarrow& V &= \tfrac{1}{2}i(\phi_1\bar{\phi}_2 - \bar{\phi}_1\phi_2) \\
V_3 &= V_1 \cdot V_2 &&\Leftrightarrow& V_3 &= V_1 V_2 \,,
\end{aligned}
\tag{7.59}
$$

and the fact that ϕ_3 is again chiral is due to the fact that $\bar{D}$ is a first-order differential operator, which

obeys the Leibniz rule,

$$\bar{D}(\phi_1\phi_2) = (\bar{D}\phi_1)\phi_2 + \phi_1\bar{D}\phi_2 = 0 \ . \tag{7.60}$$

Note that second-order conditions like $D^2\phi = 0$ are not preserved for a product.

The *kinetic superfield* is given by

$$\mathsf{T}\phi = \tfrac{1}{4}\bar{D}^2\bar{\phi} \ . \tag{7.61}$$

We see that $\mathsf{T}\phi$ is chiral, $\bar{D}(\mathsf{T}\phi) = 0$. The property $\mathsf{T}\mathsf{T} = -\square$ follows from $\overline{\mathsf{T}\phi} = \tfrac{1}{4}D^2\phi$ and

$$[\bar{D}^2, D^2] = -16\square - 8\mathrm{i}D\!\!\!/\sigma\bar{D} \ . \tag{7.62}$$

8. Superspace invariants

The attentive reader may have noticed that whereas for some of the elements of multiplet calculus simple equivalent superspace formulas could be given in the previous section, no mention was made of an equivalent of the "contragradient multiplets" or of the invariants. Indeed, the construction of invariants from superfields is the subject of the present section.

8.1. Berezin integration

The general method by which a translation invariant action is derived from fields is to integrate a Lagrangian density $L(x)$ over d^4x. The result is translationally invariant if surface terms vanish. A similar procedure can be used to construct supersymmetry invariant actions in superspace. As can be seen from eqs. (7.26), the action of Q is a derivative in θ plus a term which itself is a total divergence in x-space (and moreover parametrization dependent, cf. eq. (7.48)). Once we can define an integral $\int \mathrm{d}\theta$ in such a way that it is invariant under $\theta \rightarrow \theta + \zeta$, then the integral of any superfield over the whole of superspace $(\theta, \bar{\theta}$ and $x^\mu)$ will be invariant.

Such an integral is known. It is the *Berezin integral* [3], defined by

$$0 \equiv \int \mathrm{d}\theta; \qquad 1 \equiv \int \mathrm{d}\theta \, \theta \tag{8.1}$$

for each different θ. Since a function of any one anticommuting θ is always of the form

$$f(\theta) = f_0 + \theta f_1 \ , \tag{8.2}$$

these definitions are sufficient to define a general $\int \mathrm{d}\theta f(\theta)$. Assuming that θ is not a multiple of ζ, the translational invariance of the integral follows:

$$\int \mathrm{d}\theta f(\theta + \zeta) = \int \mathrm{d}\theta \, (f_0 + \theta f_1 + \zeta f_1) = f_1 = \int \mathrm{d}\theta f(\theta) \ . \tag{8.3}$$

Formally, differentiation and integration are the same,

$$\int d\theta\, f(\theta) = \frac{\partial}{\partial\theta} f(\theta)\,, \tag{8.4}$$

a curious fact for which we can develop an understanding by visualising power series in θ to be "modulo 2" so that raising the power (integrating) and lowering the power (differentiating) are the same thing. This also results in such strange equations as that for the δ-function:

$$\delta(\theta) = \theta; \qquad \delta(-\theta) = -\delta(\theta)\,. \tag{8.5}$$

It should be noted that any more sophisticated mathematical treatment of the Berezin integration very quickly dissolves the illusion that it may be rather trivial [53].

We define

$$\int d^2\theta \equiv \int d\theta^2\, d\theta^1; \qquad \int d^2\bar\theta \equiv \int d\bar\theta^{\dot{1}}\, d\bar\theta^{\dot{2}}\,, \tag{8.6}$$

so that

$$\int d^2\theta\, \theta^2 = \int d^2\bar\theta\, \bar\theta^2 = -2\,. \tag{8.7}$$

As mentioned before, the integral of any superfield over the whole of superspace will be an invariant,

$$\delta \int d^4x\, d^2\theta\, d^2\bar\theta\, \phi(x, \theta, \bar\theta) = 0\,, \tag{8.8}$$

provided that there is no Jacobian determinant to be considered. Actually, the functional matrix for the transformations (7.21b) is

$$\frac{\partial(x', \theta', \bar\theta')}{\partial(x, \theta, \bar\theta)} = \begin{bmatrix} \delta_\mu^\nu & -i(\sigma^\nu\bar\zeta)_\alpha & -i(\zeta\sigma^\nu)_{\dot\alpha} \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & 0 & \delta_{\dot\alpha}^{\dot\beta} \end{bmatrix} = e^X \tag{8.9}$$

with

$$X = \begin{bmatrix} 0 & -i\sigma^\nu\bar\zeta & -i\zeta\sigma^\nu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and has unit determinant, no matter how we define the determinant.

8.2. The superdeterminant ("Berezinian")

The superspace transformations (7.21b) which represent supersymmetry transformations in flat-space field theory have the functional matrix (8.9) with unit Jacobian determinant. Therefore there was no need to include a determinant in the formula (8.8) for an invariant. In the context of supergravity, however, we shall encounter general coordinate transformations of superspace and then we better have some consistent prescription for a Jacobian.

Defining a determinant for a matrix like (8.9) which has numerical entries (the δ's) as well as "anticommuting number" entries (the ζ's), is not quite straightforward. Since two ζ's do not commute, we wouldn't get the usual determinant multiplication law if we used the standard definition of the determinant. We must therefore define a new *super-determinant* (Russian authors call it the "Berezinian"; in the West it first surfaced in ref. [1]) which does indeed observe the law

$$\text{Sdet}(M_1 M_2) = (\text{Sdet } M_1)(\text{Sdet } M_2). \tag{8.10}$$

To make things easier, we start by defining a *super-trace*. Matrices like (8.9) are of the form

$$M = \begin{bmatrix} m_1 & \mu_2 \\ \mu_1 & m_2 \end{bmatrix} \quad \text{with} \quad \begin{cases} m_i & \text{commuting elements} \\ \mu_i & \text{anticommuting elements}. \end{cases} \tag{8.11}$$

Let us consider two possible definitions of a super-trace, $\text{STr } M = \text{tr } m_1 \pm \text{tr } m_2$. If we now calculate the trace of a commutator, we find it to vanish for the definition with the minus sign, but not for the usual definition with the plus sign. We can therefore maintain the very important property of any trace,

$$\text{STr}[M, N] = 0, \tag{8.12}$$

only if we define

$$\text{STr } M \equiv \text{tr } m_1 - \text{tr } m_2. \tag{8.13}$$

Equation (8.12) is very useful for our quest for a super-determinant: with

$$M \equiv \exp(\ln M) \tag{8.14}$$

we can define

$$\text{Sdet } M \equiv \exp\{\text{STr}(\ln M)\} \tag{8.15}$$

and then get

$$\begin{aligned}
\text{Sdet}(M_1 M_2) &= \text{Sdet } \exp\{\ln M_1 + \ln M_2 + \tfrac{1}{2}[\ln M_1, \ln M_2] + \cdots\} \\
&= \exp\{\text{STr}(\ln M_1 + \ln M_2)\} = \exp\{\text{STr } \ln M_1\} \exp\{\text{STr } \ln M_2\} \\
&= \text{Sdet } M_1 \text{ Sdet } M_2
\end{aligned}$$

as required. The first step followed from (7.18), with the dots standing for further multi-commutators. The second step is a consequence of (8.15) and of (8.12). Additional terms would appear if the supertrace of a commutator didn't vanish.

Equations (8.15) and (8.13) together constitute a suitable definition for the super-determinant of matrices with anticommuting off-diagonal elements.

8.3. Chiral integrals

The integral (8.8), of whose invariance we have now convinced ourselves, plays an important role because it can be used to construct invariant actions. Before we attempt to do that, we must, however, consider the special case of an integral over a chiral superfield. In that case, the shift term in (7.44) can be dropped under $\int d^4x$, and the rest is independent of $\bar{\theta}$ so that $\int d^2\bar{\theta}$ and hence the full superspace integral give zero.

On the other hand, the $\int d^4x \, d^2\theta$ alone, without the $d^2\bar{\theta}$, is already an invariant integral for chiral superfields. This is so because for them the supersymmetry algebra can be realized as coordinate transformations of the chiral subspace of superspace alone, which has coordinates x^μ and θ^α but not $\bar{\theta}^{\dot\alpha}$. This would be most obvious in the 1-parametrization. Correspondingly, we have

$$\delta \int d^4x \, d^2\theta \, \phi(x, \theta, \bar{\theta}) = 0 \qquad \text{if } \bar{D}\phi = 0$$

$$\delta \int d^4x \, d^2\bar{\theta} \, \bar{\phi}(x, \theta, \bar{\theta}) = 0 \qquad \text{if } D\bar{\phi} = 0 \, . \tag{8.16}$$

8.4. The Wess–Zumino action in superspace

Let us now try to write down an invariant action for a chiral superfield ϕ which, as we know, has ordinary field components A, B, ψ, F and G. First, as in the multiplet approach, we do some dimensional analysis: clearly each θ carries a (mass) dimension of $-\frac{1}{2}$ (so that, e.g., the exponents in eq. (7.44) are dimensionless); if we want the spinor ψ to have canonical dimension $\frac{3}{2}$, we find by looking at (7.45) that the whole superfield has dim $\phi = 1$. The integrations $d^2\theta$ and $d^2\bar{\theta}$ have dimension $+1$ each, as is clear from either (8.1) or (8.4). Thus the chiral measure $d^4x \, d^2\theta$ has dimension -3, and in order to construct a dimensionless action, the most general term involving only ϕ's and no negative-dimension coupling constants is

$$\lambda\phi + \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3 \, .$$

Clearly, an action constructed from this cannot describe dynamics: there are no time derivatives. What we need for a kinetic term is something bilinear in the superfield with either one derivative ∂_μ (dimension 1) or two derivatives D_α or $\bar{D}_{\dot\alpha}$ (dimension $\frac{1}{2}$). The first alternative cannot be realized Lorentz covariantly. In the second case we are luckier: the kinetic superfield $T\phi \equiv \frac{1}{4}\bar{D}^2\phi$, can be used to write down the Wess–Zumino action for a chiral superfield (as described in section 6, the $\lambda\phi$ term can be shifted away):

$$I = \tfrac{1}{4} \int d^4x \, d^2\theta \left(\tfrac{1}{2}\phi \, \mathsf{T}\phi - \frac{m}{2}\phi^2 - \frac{g}{3}\phi^3 \right) + \text{h.c.}$$

$$= \tfrac{1}{4} \int d^4x \, d^2\theta \left(\tfrac{1}{8}\phi \, \bar{D}^2\bar{\phi} - \frac{m}{2}\phi^2 - \frac{g}{3}\phi^3 \right) + \text{h.c.} \tag{8.17}$$

Worked out into components, this gives the same Lagrangian density as we had before, eqs. (5.2). This is clear from the relationship

$$\tfrac{1}{4} \int d^2\theta \, \phi + \text{h.c.} = [\phi]_F + 4\text{-div.} \tag{8.18a}$$

for any chiral ϕ, a formula which follows from eqs. (8.7) and (7.45). A similar relationship for the integral over the whole superspace is

$$-\tfrac{1}{2} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, V = [V]_D + 4\text{-div.} \tag{8.18b}$$

for a general superfield V.

8.5. Superfield equations of motion

The action of the Wess–Zumino model can be recast in different form. First, what is the "+ h.c." in eq. (8.17)? Well,

$$.. + \text{h.c.} = .. + \tfrac{1}{4} \int d^4x \, d^2\bar{\theta} \left(\tfrac{1}{8}\bar{\phi}D^2\phi - \frac{m}{2}\bar{\phi}^2 - \frac{g}{3}\bar{\phi}^3 \right),$$

and we concentrate on the first term of this, which we can rewrite as an integral over the whole superspace:

$$\frac{1}{32} \int d^4x \, d^2\bar{\theta} \, \bar{\phi} \, D^2\phi = \frac{1}{32} \int d^4x \, d^2\bar{\theta} \, D^2(\bar{\phi}\phi) = \frac{1}{16} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \bar{\phi}\phi .$$

This follows from the anti-chirality of $\bar{\phi}$, the equivalence of integration over and differentiation with respect to anticommuting variables and the fact that the difference between $\bar{D}$ and $-\partial/\partial\bar{\theta}$ is a derivative term which does not contribute under the d^4x-integral. Just as we have converted D^2 into $2 \, d^2\theta$, we can now convert $d^2\bar{\theta}$ into $\tfrac{1}{2}\bar{D}^2$ and get

$$= \frac{1}{32} \int d^4x \, d^2\theta \, \phi\bar{D}^2\bar{\phi},$$

i.e. just the same term again as already present in (8.17). Basically, what we have shown is that

$$\int d^4x \operatorname{Im}\left(\int d^2\theta\, \phi\, \bar{D}^2\bar{\phi}\right) = 0 .$$

The action now reads

$$I = \tfrac{1}{4}\int d^4x\, d^2\theta \left(\tfrac{1}{4}\phi\bar{D}^2\bar{\phi} - \frac{m}{2}\,\phi^2 - \frac{g}{3}\,\phi^3\right) - \tfrac{1}{4}\int d^4x\, d^2\bar{\theta}\left(\frac{m}{2}\,\bar{\phi}^2 + \frac{g}{3}\,\bar{\phi}^3\right), \tag{8.19}$$

and while in principle this is a "higher derivative action" (there is a $\bar{D}^2$ in it), we see that it contains derivatives only of $\bar{\phi}$, not of ϕ. The equation of motion for ϕ is therefore simple to calculate:

$$0 = \delta I/\delta\phi = \tfrac{1}{4}(\tfrac{1}{4}\bar{D}^2\bar{\phi} - m\phi - g\phi^2), \tag{8.20}$$

i.e. $T\phi = m\phi + g\phi^2$, the superfield form of (5.3).

Note that we were allowed to simply vary the action with respect to ϕ because we are looking at an integral over chiral superspace in which a chiral superfield is unconstrained. If, for the kinetic term, we had chosen to write

$$I_{\text{kin}} = \tfrac{1}{8}\int d^4x\, d^2\theta\, d^2\bar{\theta}\, \bar{\phi}\phi \tag{8.21}$$

then variation for ϕ would have produced the non-sensical equation $\bar{\phi} = 0$. In order to get the correct result in this way, we must impose the chiral constraint by the Lagrange multiplier:

$$I_{\text{kin}} = \tfrac{1}{8}\int d^4x\, d^2\theta\, d^2\bar{\theta}\, (\bar{\phi}\phi + \bar{D}_{\dot\alpha}\phi\bar{\Lambda}^{\dot\alpha} + \Lambda^\alpha D_\alpha\bar{\phi}). \tag{8.22}$$

This action, when varied with respect to $\bar{\Lambda}$ gives us the chiral constraint, $\bar{D}\phi = 0$, and when varied with respect to ϕ,

$$0 = \bar{D}_{\dot\alpha}\, \partial L/\partial\bar{D}_{\dot\alpha}\phi - \partial L/\partial\phi$$

gives

$$\bar{\phi} = \bar{D}_{\dot\alpha}\bar{\Lambda}^{\dot\alpha},$$

which is the general solution of the free equation of motion $\tfrac{1}{4}\bar{D}^2\bar{\phi} = 0$.

8.6. The non-renormalisation theorem

The fact that the kinetic part of the Wess–Zumino action can be written as an integral over the whole of superspace, eqs. (8.21–22), but the mass and interaction terms cannot, has important consequences. There is a theorem [30, 31] that those parts of a Lagrangian which can in principle only be written as *chiral integrals will not receive quantum corrections*.

The observed renormalisation behaviour of the Wess–Zumino model [72, 39] is a direct and predictable consequence of this: the kinetic term must be renormalised, resulting in a logarithmically divergent wave-function renormalisation, but there are no independent quadratically and linearly divergent mass and coupling constant renormalisations, respectively.

Furthermore, since the vacuum energy is strictly zero in a supersymmetric theory, we know that there will also be no contributions to the vacuum energy if supersymmetry is unbroken by renormalisation, as it is in the Wess–Zumino model [79].

9. Supersymmetric gauge theory (super-QED)

In this section, the simplest supersymmetric gauge theory will be introduced, namely super-QED [73].

Quantum electrodynamics, the gauge theory of phase transformations of the Dirac field,

$$\psi \to e^{-i\alpha(x)}\psi, \tag{9.1}$$

is a theory of matter, described by the field ψ, in interaction with radiation, described by a gauge vector field A_μ. From what we know about supersymmetry representations, see section 2, we therefore expect super-QED to describe a multiplet of matter with spins $(0, \frac{1}{2})$ in interaction with a multiplet of radiation with helicities $(\pm 1, \pm\frac{1}{2})$. Two questions arise: first, could the radiation multiplet contain helicity $\frac{3}{2}$ instead of $\frac{1}{2}$? The answer is no, since we want the model to be renormalisable. Second, could matter be the photino, the spin-$\frac{1}{2}$ partner of the photon? The answer is again no, for several reasons. Firstly, this would not allow to have massive matter for unbroken supersymmetry. Secondly, the photino can only have two physical degrees of freedom and must therefore be described by either a chiral or a Majorana spinor. In either case, the only possible gauge transformation has negative parity – for a Majorana spinor it takes the form

$$\psi \to \exp\{\gamma_5\, \alpha(x)\}\, \psi, \tag{9.2}$$

– and thus requires an axial-vector photon. Finally, neutral photons and charged matter cannot coexist in the same multiplet if the gauge transformations commute with supersymmetry transformations. If they don't, we would require a consistent description of *local* supersymmetry transformations, since then

$$[\delta_{\text{super}}, \delta_{\text{gauge}}(x\text{-dependent})] = \delta_{\text{super}}(x\text{-dependent}), \tag{9.3}$$

and consequently local translations, since

$$[\text{local supersymmetry, local supersymmetry}] = \text{local translation}. \tag{9.4}$$

"Local translations" $x^\mu \to x^\mu + \xi^\mu(x)$, on the other hand, are general coordinate transformations and lead us into the field of supergravity. We conclude:

Only in supergravity can there be gauge transformations which do not commute with supersymmetry,

and which therefore give rise to particles of different charge in the same supersymmetry multiplet.

We are thus left, as only starting point for super-QED, with having to couple whole matter multiplets (chiral multiplets) to a radiation multiplet. The spectrum we expect is that of the massive Wess–Zumino model, plus the photon and the photino. The presence of "gauginos" such as the photino, neutral massless spin-$\frac{1}{2}$ fermions, is an important prediction of supersymmetric theories.

But what is that gauge multiplet $(1, \frac{1}{2})$? What are its transformation laws, its auxiliary fields? There are several ways to tackle the problem:

(a) try finding a multiplet with a vector in it and try constructing a gauge theory around it – this was the approach in the original paper by Wess and Zumino [73];

(b) try coupling to the Noether current of the gauge transformations – this approach, first suggested by Ogievetsky and Sokatchev [48], is didactically more enlightening, and will be pursued in this section;

(c) try applying the general geometric techniques of gauge theories to superspace – this will be the subject of the next section.

9.1. The gauge current superfield

The gauge transformations of a complex four-spinor are those given by eq. (9.1), those of a Majorana spinor can only be (9.2). Since the latter are unacceptable for QED, which must give a Coulomb field of positive parity, we need at least two chiral multiplets ϕ_1 and ϕ_2 for a super-QED model. The complex spinor is then $\psi = \psi_1 + i\psi_2$, and the infinitesimal version of the gauge transformation (9.1) is $\delta_g \psi_1 = \alpha(x)\psi_2$ and $\delta_g \psi_2 = -\alpha(x)\psi_1$. Since a supersymmetry variation must commute with gauge transformations, this gauge transformation law will hold for all fields of the matter multiplets:

$$\delta_g \phi_1 = \alpha(x)\,\phi_2; \qquad \delta_g \phi_2 = -\alpha(x)\,\phi_1\,. \tag{9.5}$$

The free Wess–Zumino Lagrangian* for the two multiplets,

$$L_0 = \tfrac{1}{2}\sum_{i=1}^{2}\left[(\partial_\mu A_i)^2 + (\partial_\mu B_i)^2 + i\bar\psi_i\,\slashed\partial\psi_i + F_i^2 + G_i^2 - m(2A_iF_i + 2B_iG_i + \tfrac{1}{2}\bar\psi_i\psi_i)\right], \tag{9.6}$$

is invariant under transformations (9.5) with $\alpha = $ const. The associated Noether current of such transformations is

$$j_\mu = i\bar\psi_1\gamma_\mu\psi_2 - A_1\,\overset{\leftrightarrow}{\partial}_\mu A_2 - B_1\,\overset{\leftrightarrow}{\partial}_\mu B_2\,. \tag{9.7}$$

As a non-constant field, j_μ must be a member of a multiplet. In order to find the structure of that multiplet, we at first subject the matter fields to their (free) equations of motion. This gives us the conservation law for the Noether current,

$$\partial_\mu j^\mu = 0\,, \tag{9.8}$$

and makes it much easier to calculate the supersymmetry transformation laws for the gauge current

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* The interaction term for two multiplets cannot be made gauge invariant.

multiplet. We find that the multiplet contains, besides j^μ, the following field combinations,

$$\chi = (A_2 + \gamma_5 B_2)\,\psi_1 - (A_1 + \gamma_5 B_1)\,\psi_2$$
$$C = B_1 A_2 - A_1 B_2\,,$$
(9.9)

and that the supersymmetry transformations of these fields are exactly those of the linear multiplet, as given in eqs. (4.16).

Let us re-examine this in terms of superfields. Comparing (9.7) and (9.9) with (4.23) and (7.59), we find that the gauge current superfield is

$$J = -\tfrac{1}{2}\mathrm{i}(\phi_1\bar\phi_2 - \bar\phi_1\phi_2)\,.$$
(9.10)

The conditions for J to be a linear superfield follow from the chirality of the matter superfields, $\bar D \phi_i = 0$, and the equations of motion $\tfrac{1}{4}\bar D^2 \bar\phi_i = m\phi_i$:

$$\bar D^2 J = -\tfrac{1}{2}\mathrm{i}(\phi_1\bar D^2\bar\phi_2 - \bar D^2\bar\phi_1\phi_2) = -2\mathrm{i}m(\phi_1\phi_2 - \phi_1\phi_2) = 0$$
(9.11a)

and similarly

$$D^2 J = 0\,.$$
(9.11b)

9.2. Noether coupling and the gauge superfield

In QED, gauge invariance of the matter Lagrangian is achieved by adding a term $-j^\mu A_\mu$ to it:

$$L = \mathrm{i}\bar\psi\,\slashed\partial\psi + m\bar\psi\psi - j^\mu A_\mu\,,$$

where j^μ is the Noether current for the phase transformation on ψ. The gauge transformation law for the photon field,

$$A_\mu \to A_\mu + \partial_\mu\alpha\,,$$
(9.12)

is contragradient to the conservation law for the current, so that $j^\mu A_\mu$ transforms as a total derivative:

$$j^\mu A_\mu \to j^\mu A_\mu + \partial_\mu(j^\mu\alpha) \qquad \text{for} \qquad \partial_\mu j^\mu = 0\,.$$
(9.13)

In super-QED, the place of j^μ is taken by the superfield J as given in eq. (9.10) and that of the conservation law by the conditions (9.11). The Lagrangian will be of the form

$$L_{(1)} = L_{(0)} - j^\mu A_\mu + \text{other terms}$$
(9.14a)

or, in superfields,

$$I_{(1)} = I_{(0)} - \tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,JV$$
(9.14b)

with $I_{(0)}$ the sum of the free Wess–Zumino actions for the two superfields ϕ_1 and ϕ_2. V is a real general superfield which contains the gauge field A_μ. The numerical factor $-\frac{1}{2}$ serves to eventually reproduce the factor -1 in (9.14a); details follow from eqs. (4.25) and (8.18b).

The specific form of the gauge transformations on V can be derived from the constraints (9.11) on J: V can transform as

$$V \to V + D^2 X + \bar{D}^2 Y \equiv V + \delta_g V$$

without disturbing the second term in (9.14b) if the matter fields are on-shell and J therefore satisfies (9.11). $\bar{D}^2 Y$ is the general form of any chiral superfield, $D^2 X$ that of a (different) anti-chiral one, but if V is to be real and remain real then one must be the conjugate of the other. Thus the gauge transformations of V are

$$\delta_g V = -\tfrac{1}{2}\mathrm{i}(\Lambda - \bar{\Lambda}) \qquad \text{with } \Lambda \text{ chiral} \tag{9.15a}$$

(the factor $-\tfrac{1}{2}\mathrm{i}$ is chosen for convenience). In components, this reads

$$\begin{aligned}
&\delta_g C = B; &&\delta_g \chi = \psi \\
&\delta_g M = -F; &&\delta_g N = -G \\
&\delta_g A_\mu = \partial_\mu A; &&\delta_g \lambda = 0 \\
&\delta_g D = 0\,.
\end{aligned} \tag{9.15b}$$

We now proceed to actually check the gauge invariance of (9.14b) which so far we have only conjectured in analogy with QED, and we shall find the analogy incomplete. The full set of gauge parameters is the *chiral parameter superfield* Λ. The role of the parameter field $\alpha(x)$ which appears in (9.5) and in (9.12) is played by the A-component of Λ and the full supersymmetric gauge transformations for the matter superfields are therefore not the eqs. (9.5), which are not supercovariant as $\alpha(x)$ is not a superfield, but rather

$$\delta_g \phi_1 = \Lambda \phi_2; \qquad \delta_g \phi_2 = -\Lambda \phi_1\,. \tag{9.16}$$

For these gauge transformations, the mass term in $I_{(0)}$ is invariant but the kinetic term transforms as

$$\begin{aligned}
\delta_g I_{(0)} &= \tfrac{1}{8} \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, \delta_g(\bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2) \\
&= -\tfrac{1}{8} \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, (\bar{\phi}_2 \phi_1 - \bar{\phi}_1 \phi_2)(\Lambda - \bar{\Lambda}) = \tfrac{1}{2} \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} J \, \delta_g V\,.
\end{aligned}$$

We see that the Noether coupling term in (9.14b) would indeed cancel this if J itself were gauge invariant as it is in QED. Here, however, J transforms as

$$\delta_g J = \tfrac{1}{2}\mathrm{i}(\bar{\phi}_1\phi_1 + \bar{\phi}_2\phi_2)(\Lambda - \bar{\Lambda}) = (\bar{\phi}_1\phi_1 + \bar{\phi}_2\phi_2)\delta_g V$$

so that the variation of $I_{(1)}$ becomes

$$\delta_g I_{(1)} = -\tfrac{1}{4} \int d^4x \, d^2\theta \, d^2\bar\theta \, (\bar\phi_1 \phi_1 + \bar\phi_2 \phi_2) \, \delta_g V^2 \,.$$

Things don't seem to work quite like in QED, the first-order Noether coupling term is not sufficient to render the action gauge invariant. But then, in the presence of charged scalar matter (we have the "slepton" fields A_1, A_2, B_1 and B_2 which are the superpartners of the "lepton" ψ), we expect the Lagrangian to at least contain "sea-gull" terms $A_\mu A^\mu (A_i^2 + B_i^2)$ as in scalar electro-dynamics. Indeed, we can try to cancel the remaining $\delta_g I_{(1)}$ by adding a further term to the action,

$$I_{(2)} = I_{(1)} + \tfrac{1}{4} \int d^4x \, d^2\theta \, d^2\bar\theta \, (\bar\phi_1 \phi_1 + \bar\phi_2 \phi_2) \, V^2 \,,$$

but doing this, we now find that

$$\delta_g I_{(2)} = \int d^4x \, d^2\theta \, d^2\bar\theta \, J V^2 \, \delta_g V$$

which wants a term $\tfrac{1}{3} J V^3$ to cancel it, and so on. Clearly, in contrast to even scalar QED, the action is *non-polynomial*. A quick calculation will show that the action

$$I_{\text{kin}} = \tfrac{1}{8} \int d^4x \, d^2\theta \, d^2\bar\theta \, (\bar\phi_1 \phi_1 + \bar\phi_2 \phi_2) \cosh 2V - \tfrac{1}{4} \int d^4x \, d^2\theta \, d^2\bar\theta \, J \sinh 2V \tag{9.17}$$

is indeed gauge invariant. An elegant way of rewriting this is to define a doublet from the matter multiplets,

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_1 + i\phi_2 \\ \phi_1 - i\phi_2 \end{bmatrix}, \tag{9.18}$$

on which the gauge transformations are those of O(2),

$$\Phi \to e^{-i\Lambda} \Phi \quad \text{with} \quad \Lambda = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}. \tag{9.19}$$

Then I_{kin} becomes

$$I_{\text{kin}} = \tfrac{1}{8} \int d^4x \, d^2\theta \, d^2\bar\theta \, \bar\Phi \, e^{-2\mathbf{V}} \Phi \quad \text{with} \quad \mathbf{V} = \begin{bmatrix} V & 0 \\ 0 & -V \end{bmatrix}. \tag{9.20}$$

9.3. Wess–Zumino gauge and matter Lagrangian

In supersymmetrizing QED, we were forced to introduce two complications: first, the single real parameter field $\alpha(x)$ had to be replaced with a whole multiplet (or superfield) of parameters, Λ, and second, the Lagrangian became non-polynomial. In this subsection, we will see how the two com-

plications can be played off against each other and how we can fix the additional gauge freedoms associated with the parameter fields B, ψ, F and G in (9.15) in such a way that the Lagrangian becomes polynomial. It will still be invariant under the usual gauge transformations which have the real field $A(x)$ as parameter.

To do this, we first observe that the action (9.17) or (9.20) is non-polynomial only in the θ-$\bar{\theta}$-independent component of V. This is so because any power series in θ or $\bar{\theta}$ eventually terminates. Thus the Lagrangian will be polynomial if and only if we choose a gauge where $C = 0$. This gauge choice is available since $\delta_g C = B$, which allows to gauge away the field C completely. Clearly, such a gauge is not supersymmetric: as remarked in subsection 7.9, for unbroken supersymmetry a whole superfield will be zero if its lowest component is zero. On the other hand, the whole of V cannot be gauged away by transformations like (9.15) and thus we know that

> *a polynomial Lagrangian can only be achieved in a non-supersymmetric gauge.*

It is most convenient actually to use up all the superfluous gauge freedoms and choose a gauge where

$$C = \chi = M = N = 0 .$$
(9.21)

Such a gauge is called a *Wess–Zumino gauge*, after ref. [73]. To repeat, this gauge is *not supersymmetric*; indeed neither $\delta\chi$ nor δM or δN vanish in this gauge. The "multiplet" which remains of V in a Wess–Zumino gauge contains fields A_μ, λ and D with transformation laws

$$\delta A_\mu = i\bar{\zeta}\gamma_\mu\lambda$$
$$\delta\lambda = -i\sigma^{\mu\nu}\zeta\,\partial_\mu A_\nu - \gamma_5\zeta D$$
$$\delta D = -i\bar{\zeta}\not{\partial}\gamma_5\lambda .$$
(9.22)

These are not a representation of the supersymmetry algebra but rather one of a modified algebra [10]

$$[\delta_1, \delta_2] = 2i\bar{\zeta}_1\gamma^\mu\zeta_2\partial_\mu + \delta_g$$
(9.23a)

where δ_g is a *field dependent gauge transformation* with parameter

$$\alpha = -2i\bar{\zeta}_1\gamma^\mu\zeta_2 A_\mu .$$
(9.23b)

On the gauge field A_μ, the modified algebra takes the particularly suggestive form

$$[\delta_1, \delta_2]A_\mu = 2i\bar{\zeta}_1\gamma^\nu\zeta_2 F_{\nu\mu} .$$
(9.24)

The reason behind the modification of the superalgebra is that the Wess–Zumino gauge must be re-established after the first supersymmetry transformation. This is done by a field dependent gauge transformation on whose parameter the second supersymmetry transformation then acts.

In the Wess–Zumino gauge, the superfield V becomes

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\bar{\theta}^2\theta\lambda - i\theta^2\bar{\theta}\bar{\lambda} - \tfrac{1}{2}\theta^2\bar{\theta}^2 D$$
(9.25)

and the matter Lagrangian (9.17) contains powers of V only up to V^2 – these are the sea-gull terms – and can be evaluated as

$$L_{\text{matter}} = \tfrac{1}{2} \sum_{i=1}^{2} [\nabla_\mu A_i \nabla^\mu A_i + \nabla_\mu B_i \nabla^\mu B_i + i\bar{\psi}_i \gamma^\mu \nabla_\mu \psi_i + F_i^2 + G_i^2 - m(2A_iF + 2B_iG_i + \bar{\psi}_i\psi_i)]$$

$$- \bar{\lambda}(A_1 + \gamma_5 B_1)\psi_2 + \bar{\lambda}(A_2 + \gamma_5 B_2)\psi_1 - (A_1 B_2 - B_1 A_2)D \tag{9.30}$$

with

$$\nabla_\mu A_1 \equiv \partial_\mu A_1 - A_\mu A_2; \qquad \nabla_\mu A_2 \equiv \partial_\mu A_2 + A_\mu A_1 \tag{9.31}$$

(same for B_i and ψ_i). The contributions in the last line of (9.30) are *non-minimal coupling terms* due to the supersymmetry of the model.

9.4. The Maxwell Lagrangian

To complete the treatment of super-QED, we need a supersymmetrised Maxwell Lagrangian of the form

$$L_{\text{Maxwell}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{other terms} . \tag{9.32}$$

Since we already know that $F_{\mu\nu}$ sits in the curl-multiplet dV, see eq. (4.9), which corresponds to a chiral superfield W_α with an additional spinor index, see eq. (7.49), it is rather straightforward to construct a Maxwell action:

$$I_{\text{Maxwell}} = \frac{1}{16} \int d^4x \, d^2\theta \, W^2 + \text{h.c.} \tag{9.33}$$

All components of the curl multiplet are gauge invariant. This can be seen in superfield form from

$$\delta_g W_\alpha = \tfrac{1}{8}\bar{D}^2 D_\alpha(\Lambda - \bar{\Lambda}) = 0 . \tag{9.34}$$

The last step followed from $D\bar{\Lambda} = 0$ and $\bar{D}^2 D\Lambda = [\bar{D}^2, D]\Lambda = 4i\bar{D}\not{\partial}\Lambda = 0$.

We can expand (9.33) into components, using the explicit form (7.53) for W_α, and find

$$L_{\text{Maxwell}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}i\bar{\lambda}\not{\partial}\lambda + \tfrac{1}{2}D^2 . \tag{9.35}$$

As expected, the physical spectrum contains a photon and its superpartner, a massless neutral photino. The field $D(x)$ is auxiliary. The full Lagrangian of super-QED is now

$$L = L_{\text{Maxwell}} + L_{\text{matter}} . \tag{9.36}$$

Going back to the form (9.33) of the Maxwell action, we see that it is written as an integral over chiral superspace only. Because of the non-renormalisation theorem, the question arises whether or not

it is possible to write it as an integral over all of superspace. The answer is affirmative: from $W_\alpha = \frac{1}{4}i\bar{D}^2 D_\alpha V$ we find that we can write

$$W^2 = -\frac{1}{16}\bar{D}^2(D^\alpha V\bar{D}^2 D_\alpha V)$$

and

$$\int d^4x\, d^2\theta\, W^2 = \frac{1}{8}\int d^4x\, d^2\theta\, d^2\bar{\theta}\, VD^\alpha\bar{D}^2 D_\alpha V$$

(D^α has been partially integrated). This is Hermitian because of the identity (7.52), and we get

$$I_{\text{Maxwell}} = \frac{1}{64}\int d^4x\, d^2\theta\, d^2\bar{\theta}\, VD^\alpha\bar{D}^2 D_\alpha V. \qquad (9.37)$$

The fact that the action can be expressed as an integral over all of superspace means that we must expect a logarithmically divergent wave-function renormalisation, common to the fields of the photon and the photino.

It is possible, though not altogether straightforward, to generalise the results of this section to non-Abelian gauge theories [18, 54]. Much more, however, can be learned from a full geometric treatment of gauge theories, the subject of the next section.

10. Superspace gauge theories

The subject of this section will be the full treatment of non-Abelian gauge theory in superspace. In approach and philosophy, this is based on papers by Wess and Zumino [74, 75] (see also [59]).

10.1. Ordinary gauge theories

Ordinary gauge theories in x-space are based on some unitary Lie group with algebra

$$[T_m, T_n] = ic_{mn}{}^l T_l; \qquad T_m = (T_m)^\dagger \qquad (10.1)$$

and group elements

$$g = \exp(i\alpha^m T_m) \qquad (10.2)$$

which are defined by real parameters $\alpha^m = (\alpha^m)^*$. The gauge group is represented on quantum fields by unitary transformations

$$U(g_1)\, U(g_2) = U(g_1 \circ g_2) \qquad (10.3)$$

whose effect, in the case of *gauge covariant fields*, is a reshuffling of field components:

$$\phi_i \to \phi_i' \equiv U(g) \, \phi_i \, U^{-1}(g) = [r^{-1}(g)]_i{}^j \, \phi_j = (e^{-i\alpha}\phi)_i \,, \tag{10.4a}$$

or, infinitesimally,

$$\delta_g \phi = -i\alpha\phi \,. \tag{10.4b}$$

We have written the matrix representation $r(g)$ of the group as

$$r(g) = e^{i\alpha} \,, \tag{10.5}$$

where we understand the α without any further indices to be a "Lie-algebra valued" parameter,

$$\alpha = \alpha^m \Sigma(T_m) \,, \tag{10.6}$$

with $\Sigma(T)$ a matrix representation of the Lie-algebra (10.1).

In the case of theories with *local gauge invariance* (the interesting ones), the parameters are space–time dependent:

$$\alpha = \alpha(x) \,. \tag{10.7}$$

Therefore the gradient of a covariant field is not covariant,

$$U(\partial_\mu\phi)U^{-1} = \partial_\mu(U\phi U^{-1}) = e^{-i\alpha} \, \partial_\mu\phi + (\partial_\mu \, e^{-i\alpha})\phi \,,$$

and in order to mend this defect, we must introduce a *gauge connection* (gauge "potential") field A_μ which is also Lie-algebra valued,

$$A_\mu = A_\mu{}^m \, \Sigma(T_m) \,, \tag{10.8}$$

and which has an inhomogeneous transformation law

$$UA_\mu U^{-1} = e^{-i\alpha}(A_\mu - i\partial_\mu) e^{i\alpha} \tag{10.9a}$$

or, infinitesimally,

$$\delta_g A_\mu = \partial_\mu\alpha + i[A_\mu, \alpha] \,. \tag{10.9b}$$

The inhomogeneous term in δA_μ can cancel the $\partial_\mu\alpha$-term in $\delta_g\phi$, and we get

$$U(\partial_\mu\phi + iA_\mu\phi)U^{-1} = e^{-i\alpha} \, (\partial_\mu\phi + iA_\mu\phi) \tag{10.10}$$

so that

$$\nabla_\mu \equiv \partial_\mu + iA_\mu \tag{10.11}$$

is a *gauge-covariant derivative*. From the commutator of two ∇'s, we get the *gauge curvature* ("field

strength") $F_{\mu\nu}$,

$$[\nabla_\mu, \nabla_\nu]\phi = [\partial_\mu, \partial_\nu]\phi + iF_{\mu\nu}\phi = iF_{\mu\nu}\phi ,$$ (10.12)

another Lie-algebra valued field. It is given in terms of the A_μ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$ (10.13)

and transforms covariantly in the adjoint representation of the gauge group. In our conventions, this transformation law reads

$$UF_{\mu\nu}U^{-1} = e^{-i\alpha}F_{\mu\nu}\, e^{i\alpha} ; \qquad \delta_g F_{\mu\nu} = i[F_{\mu\nu}, \alpha] .$$ (10.14)

Equation (10.12) is sometimes called the "Ricci-identity", in analogy with the corresponding case in general relativity. Similarly, the cyclic identity

$$\nabla_{[\kappa}F_{\mu\nu]} = 0$$ (10.15)

is sometimes called "Bianchi-identity". It follows from the Jacobi-identity for three ∇'s and from (10.12). In the case of an Abelian gauge group, it reduces to the homogeneous Maxwell equations

$$\varepsilon^{\mu\nu\rho\sigma}\, \partial_\nu F_{\rho\sigma} = 0 .$$

10.2. Minimal superspace gauge transformations?

The original papers by Wess and Zumino [73] on super-QED and by Ferrara and Zumino [18] and Salam and Strathdee [54] on non-Abelian supersymmetric gauge theories suggest that it should be possible to write a superspace gauge theory which involves only a single real superfield V as connection and a chiral superfield Λ as gauge parameter. Such a scheme is minimal: a single field $\alpha(x)$ cannot be a supersymmetric local gauge parameter since it cannot be a superfield unless it is a constant. The gauge parameter must therefore be a superfield, and the smallest one available is the chiral superfield.

As we shall see, the generalisation to superspace of the geometric approach outlined so far in this section will not give us such a minimal scheme (we could, however, have got the minimal scheme by suitably adapting the Noether coupling method to the non-Abelian case). For the time being, let us nevertheless pursue.

10.3. The maximal approach

For each of the superspace derivatives ∂_μ, D_α and $\bar{D}_{\dot\alpha}$ we introduce a corresponding gauge potential superfield $A_\mu(x, \theta, \bar\theta)$, $A_\alpha(x, \theta, \bar\theta)$ and $A_{\dot\alpha}(x, \theta, \bar\theta)$. As gauge parameter, we take a general superfield $X(x, \theta, \bar\theta)$. We do not as yet make reality assumptions about any of these, because we know that the minimal scheme involves a chiral superfield as gauge parameter, which is intrinsically complex. In particular, $A_\mu \neq (A_\mu)^\dagger$, $A_{\dot\alpha} \neq (A_\alpha)^\dagger$ and $X \neq X^*$.

To simplify notation, I use capital Latin indices to denote the complete set of superspace-indices: $A = (\mu, \alpha, \dot\alpha)$. Thus:

$$D_A = (\partial_\mu, D_\alpha, \bar{D}_{\dot{\alpha}})$$

$$A_A = (A_\mu, A_\alpha, A_{\dot{\alpha}}).$$ (10.16)

The gauge connections transform in analogy with eq. (10.9) as

$$A_A \to e^{-iX}(A_A - iD_A) e^{iX}$$ (10.17)

and we can define covariant derivatives as before

$$\nabla_A \equiv D_A + iA_A.$$ (10.18)

In contrast to ordinary space, where $[\partial_\mu, \partial_\nu] = 0$, we have now

$$[D_A, D_B\} = iT_{AB}{}^C D_C,$$ (10.19a)

namely the algebra (7.41) of the spinor derivatives, with

$$T_{\alpha\beta}{}^\mu = T_{\beta\alpha}{}^\mu = 2\sigma^\mu{}_{\alpha\beta} \quad \text{and all other } T\text{'s zero}.$$ (10.19b)

The "graded commutator" $[..,..\}$ was introduced in subsection 2.2. The non-Abelian structure of (10.19) modifies the Ricci identity

$$[\nabla_A, \nabla_B\} = iT_{AB}{}^C \nabla_C + iF_{AB}$$ (10.20)

and the definition of F_{AB} in terms of A_A:

$$F_{AB} = D_A A_B \mp D_B A_A + i[A_A, A_B\} - iT_{AB}{}^C A_C.$$ (10.21)

With this definition, including the last term, F is covariant,

$$X_{AB} \to e^{-iX}F_{AB} e^{iX},$$ (10.22)

and fulfills Bianchi identities. These are, however, also modified by the presence of the "torsion" term in (10.19):

$$\sum_{[ABC\}} [\nabla_A F_{BC} - iT_{AB}{}^D F_{DC}] = 0.$$ (10.23)

In extending the gauge theory set-up to superspace, we had to introduce the complex connection superfields A_A which between them have 16 times as many components as we expect for the minimal version with just one real V. The complex parameter superfield X has four times as many components as the chiral Λ.

Something drastic must happen to get us back to the minimal scheme. This cannot be a partial gauge choice because X has not enough components to gauge away so many superfluous ones in A_A. What we need are super- and gauge-covariant constraints on the theory.

10.4. Constraints

Such constraints must set covariant superfields to zero. Covariant, because they should not break gauge invariance; superfields, because they should not break supersymmetry. Clearly, the F_{AB} are suitable candidates for being constrained.

For an ordinary gauge theory this cannot be done. The only F is $F_{\mu\nu}$, which contains Lorentz representations

$$F_{\mu\nu}: \quad (1,0) + (0,1)$$

which are linked to each other by the Bianchi identity. Correspondingly, setting one of them to zero, say

$$0 = F_{\mu\nu} + \tfrac{1}{2} i \, \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},$$

gives the equation of motion $0 = \nabla^\mu F_{\mu\nu}$. The lesson we learn is to be careful not to over-constrain the system, as we may end up with on-shell conditions for the fields, or even with a trivial theory, $F_{\mu\nu} = 0$.

For superspace gauge theory, the scope is much wider. Indeed, decomposing F_{AB} into Lorentz representations, we find

$$F_{AB}: \quad 2 \cdot (1,0) + 2 \cdot (0,1) + (1,\tfrac{1}{2}) + (\tfrac{1}{2},1) + (\tfrac{1}{2},0) + (0,\tfrac{1}{2}) + (\tfrac{1}{2},\tfrac{1}{2}) \tag{10.24}$$

which leaves plenty of choice for constraints.

Which constraints to impose is a highly non-trivial decision which lies at the heart of trying to find superspace versions of supersymmetric theories. The sizable confusion which originally dominated this point has been considerably reduced by a classification for constraints given by Gates, Stelle and West [26]. We will follow their argumentation.

First there are what is called *conventional constraints**: the transformation law for a gauge potential, eq. (10.17), remains unchanged if we add to it a covariant field in the adjoint representation:

$$A_A \to A_A + F_A \,.$$

We could, e.g., add the superfield

$$F_\mu \equiv -\tfrac{1}{4} i \bar\sigma_\mu{}^{\beta\alpha} \, F_{\alpha\beta}$$

to A_μ and get a new A_μ. For this new, redefined A_μ^{new} we then get $F_{\alpha\beta}^{\text{new}} = 0$. Thus,

$$F_{\alpha\beta} = 0 \tag{10.25}$$

is a "safe" constraint which can always be imposed. It just redefines A_μ suitably. Alternatively, we could say that (10.25) can always be solved to express A_μ in terms of A_α and $A_{\dot\alpha}$:

$$A_\mu = -\tfrac{1}{4} i \bar\sigma_\mu{}^{\beta\alpha} \left(D_\alpha \, A_\beta + \bar D_\beta \, A_\alpha + i\{A_\alpha, A_\beta\} \right). \tag{10.26}$$

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* A bad name – "redefinition constraints" would be better, since "conventional" can also mean "ordinary, unimaginative" – which these are not!

Note that all of this only works because the "torsion" in (10.21) has an inverse [63]. Such a constraint which can be solved for one of the fields in the theory is a conventional constraint.

The second type of constraints which we need are called *representation preserving constraints*. These come about through a somewhat more "physical" argumentation: the superfields which describe scalar and spinor matter are chiral or anti-chiral. Eventually, we will want to couple them to gauge fields in order to have a supersymmetric extension of, say, quantum chromodynamics. As we have already seen, derivatives must be gauge covariantised, and correspondingly we expect a "gauge-chiral" superfield to fulfill a constraint of the sort

$$\nabla_{\dot{\alpha}}\phi = 0,\tag{10.27a}$$

and its conjugate, the "gauge-anti-chiral" superfield, should fulfill

$$\nabla_{\alpha}\bar{\phi} = 0.\tag{10.27b}$$

Such differential equations have integrability conditions. The equations are only consistent for all ϕ if

$$\{\nabla_{\alpha}, \nabla_{\beta}\} = iF_{\alpha\beta} = 0, \qquad \{\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}\} = iF_{\dot{\alpha}\dot{\beta}} = 0.\tag{10.28}$$

The constraints (10.28) are called "representation preserving" because they are necessary in order to ensure that the special supersymmetry representations "chiral superfield" and "anti-chiral superfield" survive in the presence of non-zero gauge coupling when there is a term with a gauge potential present in the conditions (10.27).

One more type of constraint will be necessary in order to boil down our scheme to the minimal one: a *reality constraint*. In ordinary gauge theories, a real gauge parameter naturally leads to a real gauge potential A_{μ}. An imaginary part of A_{μ} would just be a covariant field in the adjoint representation of the gauge group. Here, where we a-priori allow complex gauge parameter superfields, we must explicitly ensure that there is only one gauge field, not real and imaginary parts separately. We therefore impose the soft reality constraint that A_{μ} and $(A_{\mu})^{\dagger}$ be gauge equivalent,

$$A_{\mu} = (A_{\mu})^{\dagger} + \text{gauge-transformation}.\tag{10.29}$$

We collect our constraints,

$$F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = F_{\alpha\beta} = 0; \qquad \text{Im } A_{\mu} = \text{pure gauge}\tag{10.30}$$

and go about solving them.

"Solving a constraint" means expressing a field which is subject to some differential condition in terms of derivatives of other fields in such a way that the properties of the derivatives ensure that the original condition holds. A good example of this is provided in elementary electro-dynamics lectures, where the vector potential is usually introduced as a solution of the constraints which the homogeneous Maxwell equations $\varepsilon^{\mu\nu\rho\sigma} \partial_{\nu}F_{\rho\sigma} = 0$ impose on the electric and magnetic field strengths. The curl of a potential, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, is a solution because partial differentiations are assumed to commute with each other. In our case of superspace gauge theory, we have this already built in: we started out with

potentials rather than field strengths, so that the Bianchi identities (10.23), one of which is the homogeneous Maxwell equation, are really identities. If we had interpreted the identities as "constraints" on the F's, then eq. (10.21) would have been the solution in terms of potentials A_A. Equations (10.30), however, are true constraints: not every set of potentials will satisfy them.

10.5. Prepotentials

We do, of course, already know what is necessary to solve the conventional constraint (10.25): A_μ cannot be independent any more, it must be expressed in terms of differentiations acting on A_α and $A_{\dot\alpha}$, see eq. (10.26). The representation preserving constraints (10.28) are a bit more difficult to solve: we observe that, in some vague sense, the "space is spinorially flat", i.e. that all F's vanish which have both legs in spinorial directions. It is therefore not surprising that the solution is "spinorially pure gauge". This is jargon for

$$A_\alpha = -i\, e^{2V} D_\alpha\, e^{-2V}$$
$$A_{\dot\alpha} = -i\, e^{2U} \bar{D}_{\dot\alpha}\, e^{-2U}, \tag{10.31}$$

which is a solution of (10.28) for general complex superfields U and V.

It may look as if we have done too much now: both A_α and $A_{\dot\alpha}$ are "pure gauge" and A_μ is expressed in terms of them. So can we gauge away everything? The answer is, of course, no. Since U and V are different, and since we have only one gauge parameter superfield X available, we can at most gauge away one of them. Before doing so, let us have a look at the gauge transformation properties of V and U. In order to satisfy (10.17), the original gauge transformations (parameter X) must take the form

$$e^{-2V} \to e^{-2V}\, e^{iX}, \qquad e^{-2U} \to e^{-2U}\, e^{iX}. \tag{10.32}$$

The solutions (10.31) of the constraints express the gauge potentials in terms of the *prepotentials* U and V. These are called "prepotentials" because they are unconstrained superfields in terms of which the potentials are expressed in such a way that the constraints are solved. Just as one can look upon gauge transformations of the vector potential as having arisen out of solving the Bianchi constraint on the field strength in terms of the potential, so solving the constraints on A_A in terms of prepotentials introduces *additional* transformations beyond (10.32) under which the A_A are invariant. These are

$$e^{-2V} \to e^{-i\bar\Lambda'}\, e^{-2V}, \qquad e^{-2U} \to e^{-i\Lambda}\, e^{-2U} \tag{10.33}$$

with Λ chiral and $\bar\Lambda'$ anti-chiral.

10.6. Supersymmetric gauge choice

At this stage, one usually makes a partial gauge choice. It may seem somewhat odd, but the most useful one is to gauge $A_{\dot\alpha}$ to zero. This can be done by choosing $iX = 2U$ in (10.32). We then have

$$A_{\dot\alpha} = 0 \tag{10.34a}$$

$$\nabla_{\dot\alpha} = \bar{D}_{\dot\alpha} \tag{10.34b}$$

$$A_\mu = -\tfrac{1}{4}\bar\sigma_\mu{}^{\dot\beta\alpha}\, \bar{D}_{\dot\beta}(e^{2V}D_\alpha\, e^{-2V}) \tag{10.35}$$

and the only remains of the original gauge parameter is a chiral piece, $X = \Lambda$, which leaves the condition $A_{\dot\alpha} = 0$ invariant.

The remaining gauge freedom on V is now

$$e^{-2V} \to e^{-i\bar\Lambda'}\, e^{-2V}\, e^{i\Lambda}\, . \tag{10.36}$$

The gauge choice $A_{\dot\alpha} = 0$ is supersymmetric, since a superfield has been gauged away and a superfield's worth of gauge parameters have been fixed. Also, the remaining gauge freedom (10.36) is expressed in terms of superfields.

10.7. Solving the reality constraint

In order to solve (10.29), we first consider the Abelian case. There, eq. (10.35) reduces to

$$A_\mu = \tfrac{1}{2}\bar{D}\bar\sigma_\mu\, DV$$

and the constraint has the solution

$$V = V^\dagger + \text{chiral} + \text{anti-chiral}$$

because "chiral + anti-chiral" is the solution to $\bar{D}\bar\sigma_\mu D \propto \partial_\mu$. We can now make one last gauge choice, and choose $\Lambda + \bar\Lambda'$ such that strictly

$$V = V^\dagger \tag{10.37}$$

which restricts (10.36) to $\bar\Lambda' = \bar\Lambda$. This argumentation carries through to the non-Abelian case as well. Although it is then not possible to find a supersymmetric gauge where $A_\mu = (A_\mu)^\dagger$, we can still have (10.37) and gauge transformations

$$e^{-2V} \to e^{-i\bar\Lambda}\, e^{-2V}\, e^{i\Lambda} \tag{10.38}$$

which preserve it. Then A_μ is gauge-equivalent to $A_\mu^\dagger$.

We have now succeeded, by imposing the constraints (10.30) and the two gauge choices (10.34) and (10.37), in reducing everything to the desired minimal theory, containing one real prepotential superfield V and one chiral gauge parameter Λ. In the Abelian case, the theory is identical to that described in section 8.

10.8. The field strength superfield and the Yang–Mills action

The lowest dimensional surviving field strength superfields are $F_{\mu\alpha}$ and $F_{\mu\dot\alpha}$. We can either invoke the Bianchi identities [59], eqs. (10.23), or the explicit solution to show that the constraints imply that

these are of the form

$$
\begin{aligned}
&F_{\mu\dot\alpha} = -i\bar\sigma_{\mu\dot\alpha}{}^\beta W_\beta; \qquad F_{\mu\alpha} = -i\sigma_{\mu\alpha\dot\beta}\,\bar W^{\dot\beta} \\
&W_\alpha = \tfrac{1}{8}\bar D^2 A_\alpha \qquad\qquad \bar W_{\dot\alpha} = (W_\alpha)^\dagger
\end{aligned}
\tag{10.39}
$$

where W_α is obviously chiral and gauge covariant. Also from the Bianchi identities, one can show that

$$
\nabla^\alpha W_\alpha + \nabla_{\dot\alpha}\bar W^{\dot\alpha} = 0.
\tag{10.40}
$$

These conditions restrict W_α to contain
  one chiral spinor λ_α,
  one real pseudoscalar D,
  one antisymmetric tensor $F_{\mu\nu}$, subject to $\nabla_{[\mu}F_{\kappa\lambda]} = 0$,
all in the adjoint representation of the gauge group.

Since W_α is gauge-covariant (in the adjoint representation) rather than gauge-invariant (as it was in the Abelian case), its explicit form will be gauge dependent. In the Wess–Zumino gauge it is

$$
W = e^{-i\theta\sigma\bar\theta}[\lambda - \tfrac{1}{2}i\sigma^{\mu\nu}\theta F_{\mu\nu} + i\theta D - i\theta^2\sigma^\mu \nabla_\mu\bar\lambda]
\tag{10.41}
$$

with $F_{\mu\nu}$ the non-Abelian field strength (10.13) and the covariant derivative of the "gaugino" field λ that for the adjoint representation of the gauge group:

$$
\nabla_\mu\lambda = \partial_\mu\lambda + i[A_\mu, \lambda].
\tag{10.42}
$$

The supersymmetry transformations which one derives from this by naive application of the superspace formulas (7.25–26) are, in 4-spinor notation,

$$
\begin{aligned}
&\delta\lambda = -\tfrac{1}{2}i\sigma^{\mu\nu}\zeta F_{\mu\nu} - \gamma_5\zeta D \\
&\delta F_{\mu\nu} = -i\bar\zeta(\gamma_\mu\nabla_\nu - \gamma_\nu\nabla_\mu)\lambda \\
&\delta D = -i\zeta\gamma^\mu\gamma_5\nabla_\mu\lambda.
\end{aligned}
\tag{10.43a}
$$

These involve the typical non-linear terms of non-Abelian gauge theories, and their algebra is that of "supersymmetry modulo gauge transformations" [10], namely that of eqs. (9.23). This is of course due to having chosen the Wess–Zumino gauge. In contrast to the Abelian case, there is no clear advantage in using W_α instead of V itself – both are gauge dependent. Correspondingly, we can replace $\delta F_{\mu\nu}$ in the multiplet (10.43) by

$$
\delta A_\mu = i\bar\zeta\gamma_\mu\lambda
\tag{10.43b}
$$

without changing the algebra.

The action for the gauge fields is now very similar to the Abelian case,

$$
I_{\rm YM} = \tfrac{1}{16}\int d^4x\, d^2\theta\, {\rm tr}\, W^2 + {\rm h.c.},
\tag{10.44}
$$

which is supersymmetric and gauge-invariant. If we work this out into components, we find

$$L_{YM} = \text{tr}(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}i\bar{\lambda}\gamma^\mu\nabla_\mu\lambda + \tfrac{1}{2}D^2).$$

(10.45)

The equations of motion are

$$\nabla^\nu F_{\mu\nu} = \tfrac{1}{2}\{\bar{\lambda}, \gamma_\mu\lambda\}$$

$$i\gamma^\mu\nabla_\mu\lambda = 0, \qquad D = 0$$

(10.46)

in components and

$$\nabla^\alpha W_\alpha - \nabla_{\dot\alpha}\bar{W}^{\dot\alpha} = 0$$

(10.47)

in superfield language (note the sign change between the constraint (10.40), which is one of the Bianchi identities, and the equation of motion).

The action (10.44) can also be written as an integral over the whole of superspace, along the lines discussed at the end of the previous section. We therefore expect infinite wave-function renormalisation.

10.9. The matter Lagrangian

In order to gauge covariantise the kinetic term (8.19) of the Wess–Zumino action, it will not be sufficient to just make minimal substitutions $D_A \to \nabla_A$, because $\bar{\phi}\phi$ by itself is not invariant:

$$\phi \to e^{-i\Lambda}\phi, \qquad \bar{\phi} \to \bar{\phi}\, e^{i\bar{\Lambda}}$$

$$\bar{\phi}\phi \to \bar{\phi}\, e^{i\bar{\Lambda}}\, e^{-i\Lambda}\phi \neq \bar{\phi}\phi.$$

(10.48)

It is clear, however, that the transformations of e^{-2V}, eq. (10.38), are such that $\bar{\phi}\, e^{-2V}\phi$ is invariant, and we can write the kinetic part of the matter action as

$$I_{kin} = \tfrac{1}{8}\int d^4x\, d^2\theta\, d^2\bar{\theta}\, \bar{\phi}\, e^{-2V}\, \phi.$$

(10.49)

The corresponding formula (9.20) for the Abelian case was explicitly derived at the end of subsection 9.2. The term e^{-2V} has a geometric meaning: whereas the chiral superfield ϕ is also gauge-chiral, due to the gauge choice (10.34), its conjugate $\bar{\phi}$ is anti-chiral but not gauge-anti-chiral. For $\bar{\phi}\, e^{-2V}$, however, we have

$$\nabla_\alpha(\bar{\phi}\, e^{-2V}) = (D_\alpha\bar{\phi} - i\bar{\phi}A_\alpha)\, e^{-2V} + \bar{\phi}(D_\alpha\, e^{-2V} + i[A_\alpha, e^{-2V}])$$

$$= \bar{\phi}D_\alpha\, e^{-2V} - i\bar{\phi}\, e^{-2V}A_\alpha$$

$$= \bar{\phi}D_\alpha\, e^{-2V} - \bar{\phi}\, e^{-2V}(e^{2V}D_\alpha\, e^{-2V}) = 0$$

and $\bar{\phi}\, e^{-2V}$ is gauge-anti-chiral, though not anti-chiral.

As in the Abelian case, we can go to the Wess–Zumino gauge to evaluate (10.49) in components. Unless the matter is in a real representation of the gauge group, it would be very inconvenient to work in a basis of real matter fields; the following Lagrangian is therefore given in two-component notation, with complex fields A and F as they appear in the expansion (7.45) of the chiral multiplet:

$$L_{\text{kin}} = \tfrac{1}{2}\nabla_\mu A^\dagger \nabla^\mu A + i\bar\psi\bar\sigma^\mu \nabla_\mu \psi + \tfrac{1}{2}F^\dagger F + iA^\dagger \lambda\psi - i\bar\psi\bar\lambda A + \tfrac{1}{2}A^\dagger DA .$$
(10.50)

A special case is matter in the adjoint representation of the gauge group. Then it makes sense to use real four-component notation and the Lagrangian can be cast into the form

$$L_{\text{kin}} = \text{tr}[\tfrac{1}{2}\nabla_\mu A\nabla^\mu A + \tfrac{1}{2}\nabla_\mu B\nabla^\mu B + \tfrac{1}{2}i\bar\psi\gamma^\mu \nabla_\mu \psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2$$

$$- iA[B, D] - i\bar\psi[\lambda, A] - i\bar\psi\gamma_5[\lambda, B]] .$$
(10.51)

10.10. Supersymmetric Yang–Mills theory (general case)

The most general $N = 1$ supersymmetric Yang–Mills theory is not just the sum of the Lagrangians (10.45) and (10.50) but can also involve self-interaction terms similar to those in the Wess–Zumino model as long as these are gauge invariant. The most general superpotential would be

$$V = b^a \phi_a + \tfrac{1}{2}m^{ab}\phi_a\phi_b + \tfrac{1}{3}g^{abc}\phi_a\phi_b\phi_c .$$
(10.52)

The indices are taken from the representation of the gauge group under which ϕ transforms, the potential is gauge invariant if the parameters b, m and g are numerically invariant symmetric tensors of that representation. This representation need, of course, not be irreducible, and there is wide scope for different such models.

The full component Lagrangian now looks as follows (in 2-spinor notation):

$$L = \text{tr}[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu \nabla_\mu \bar\lambda + \tfrac{1}{2}D^2] + \tfrac{1}{2}\nabla_\mu A_a \nabla^\mu A^{\dagger a} + i\psi_a\sigma^\mu \nabla_\mu \bar\psi^a + \tfrac{1}{2}F_a F^{\dagger a}$$

$$+ iA^{\dagger a}\lambda_a{}^b \psi_b - i\bar\psi^a \bar\lambda_a{}^b A_b + \tfrac{1}{2}A^{\dagger a}D_a{}^b A_b$$

$$- \tfrac{1}{2}[b^a F_a + m^{ab}(F_a A_b + \psi_a\psi_b) + g^{abc}(A_a A_b F_c + 2\psi_a\psi_b A_c) + \text{h.c.}] .$$
(10.53)

This is the most general renormalisable supersymmetric Lagrangian, except for a possible Fayet–Iliopoulos term which we shall get to know in the next section. Upon elimination of the auxiliary fields all terms which contain either F_a or $F^{\dagger a}$ or $D_a{}^b$ are replaced by

$$-U = -\tfrac{1}{2}F_a F^{\dagger a} - \tfrac{1}{2}D_m D_m$$
(10.54a)

with

$$F^{\dagger a} = m^{ab}A_b + g^{abc}A_b A_c$$
$$D_m = -\tfrac{1}{2}A^{\dagger a}\Sigma_a{}^b(T_m)A_b$$
(10.54b)

where $\Sigma_a{}^b(T_m)$ are the representation matrix elements of the generators of the gauge group.

11. Spontaneous supersymmetry breaking (II)

The presence of the pseudo-scalar auxiliary field $D(x)$ in the gauge multiplet will have its effect on the scalar potential and thus on the spontaneous breaking of the symmetries of the model [13]. Since, in particular, the value of the potential at its minimum determines whether or not supersymmetry is spontaneously broken, we must re-examine the possibilities for spontaneous supersymmetry breaking in the presence of gauge interactions. The terms in the Lagrangians (10.45) and (10.50) which can contribute to the potential are

$$L_{pot} = \tfrac{1}{2}D^2 + \tfrac{1}{2}DA^\dagger \tau A + \tfrac{1}{2}F^\dagger F. \tag{11.1}$$

In this notation, the adjoint representation of the gauge group is denoted by vectors (bold italics) and the matrices τ are the generators of the representation which acts on the matter fields. The equations of motion for the auxiliary fields are

$$D = -\tfrac{1}{2}A^\dagger \tau A; \qquad F = 0 \tag{11.2}$$

and putting this back into $U = -L_{pot}$, we find

$$U = -L_{pot} = \tfrac{1}{2}D^2 + \tfrac{1}{2}|F|^2 \tag{11.3}$$

with D and F now short-hand for the functions D(other fields) and F(other fields) which are given by the equations of motion, here by eqs. (11.2). The potential of a "minimal" gauge model, as described by the Lagrangians (10.45) and (10.50), has its minimum at $\langle A \rangle = \langle D \rangle = \langle F \rangle = 0$ and hence at $\langle U \rangle = 0$. Neither gauge nor supersymmetry invariance are broken, because the conditions for symmetry breaking are

 (a) gauge invariance is spontaneously broken if the vacuum expectation value is not zero for a field which is not gauge-invariant;

 (b) supersymmetry is spontaneously broken if the vacuum expectation value of the energy, i.e. of the potential, is not zero.

In the more complicated model (10.53) which involves self-coupling terms for matter multiplets we may get the O'Raifeartaigh type of supersymmetry breaking [50], provided that one of the multiplets is uncharged so that a bF term is gauge-invariant and can be present to trigger the symmetry breaking. As we recall from section 6, we get spontaneous supersymmetry breaking when such a term (linear in an auxiliary field) cannot be shifted away.

11.1. The Fayet–Iliopoulos term

If the gauge group contains an Abelian factor, a similar term (linear in an auxiliary field) can be added to the Lagrangian, the "Fayet–Iliopoulos term", named after ref. [13],

$$L_{F.I.} = \xi D, \tag{11.4a}$$

without breaking gauge invariance. At the moment we do not worry about possible parity-violation by such a term (D is pseudo-scalar if A_μ is a polar vector). Since the D's are in the adjoint representation,

the Fayet–Iliopoulos term is gauge-invariant only for an Abelian gauge group. It is, of course, supersymmetric since D transforms into a derivative. In superspace notation, the contribution to the action is

$$I_{\text{F.I.}} = -\tfrac{1}{2} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \xi V. \tag{11.4b}$$

To study the effect of such a term, we can consider the *simplest Abelian gauge model* [44] where a single chiral superfield undergoes phase transformations

$$\phi \to e^{-ig\Lambda} \phi. \tag{11.5}$$

In contrast to the preceding two sections, the gauge coupling constant g is here explicitly written in order to show its effect on the symmetry breaking mechanisms.

Since there is only one Majorana spinor present in a single chiral multiplet, the gauge transformation on it must take the form of γ_5-transformations $\psi \to \exp\{ig\gamma_5\alpha(x)\} \psi$. This assigns negative parity to the charge which was why super-QED had to involve two chiral multiplets, see section 9.

The total superfield action for the model is

$$I = \left[\tfrac{1}{16} \int d^4x \, d^2\theta \, W^2 + \text{h.c.}\right] + \tfrac{1}{8} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, (\bar{\phi} \, e^{-2gV} \phi - 4\xi V) \tag{11.6}$$

and the terms in the Lagrangian which are relevant for the potential are

$$L_{\text{pot}} = \tfrac{1}{2} D^2 + \frac{g}{2} A^\dagger DA + \tfrac{1}{2} F^\dagger F + \xi D \ ; \tag{11.7}$$

a mass term is not possible since $m\phi^2$ is not invariant under (11.5). The equations of motion for the auxiliary fields are

$$D = -\frac{g}{2} A^\dagger A - \xi \qquad \text{and} \qquad F = 0 \tag{11.8}$$

and the potential is

$$U = \tfrac{1}{2} D^2 + \tfrac{1}{2}|F|^2 = \frac{1}{2}\left(\frac{g}{2} A^\dagger A + \xi\right)^2 . \tag{11.9}$$

After elimination of the auxiliary fields, the "on-shell" Lagrangian becomes

$$L = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\lambda} \slashed{\partial} \lambda + \tfrac{1}{2} \nabla_\mu A^\dagger \nabla^\mu A + \frac{i}{2} \bar{\psi}\gamma^\mu \nabla_\mu \psi + ig\bar{\lambda}\frac{-1+i\gamma_5}{2}\psi A^\dagger - ig\bar{\lambda}\frac{-1-i\gamma_5}{2}\psi A - U \tag{11.10}$$

with $\nabla_\mu A \equiv \partial_\mu A + igAA_\mu$ and $\nabla_\mu\psi \equiv \partial_\mu\psi - g\gamma_5\psi A_\mu$ and U given in (11.9).
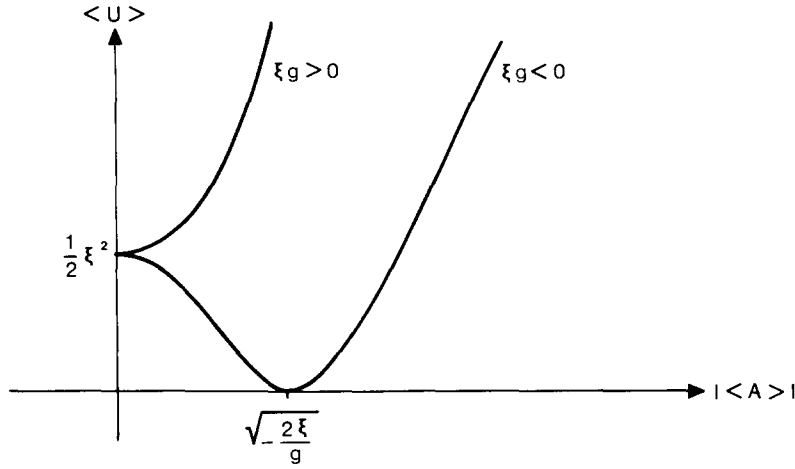
Fig. 11.1. The potential for the simplest gauge model.

If we plot $\langle U \rangle$ against $|\langle A \rangle|$ we find two very different types of potential, depending on whether $\xi g < 0$ or $\xi g > 0$ (see fig. 11.1).

11.2. The supersymmetric Higgs model

The first case which we consider, that of $\xi g < 0$, has the minimum of the potential at

$$\langle U \rangle = 0 \quad \text{and} \quad \langle A \rangle = \sqrt{-2\xi/g} \equiv m/g. \tag{11.11}$$

We therefore expect supersymmetry to be preserved but gauge invariance to be spontaneously broken. This will turn out to be the supersymmetric version of the Higgs model [15].

The usual Higgs model substitutions in the Lagrangian (11.10), with the parameter m as defined by (11.11),

$$A = \left(\frac{m}{g} + C\right) \exp(\mathrm{i}g\Theta/m), \qquad A_\mu = B_\mu - \frac{1}{m} \partial_\mu \Theta, \tag{11.12a}$$

together with the more unusual substitution

$$\psi = \exp(-g\gamma_5\Theta/m)\chi, \tag{11.12b}$$

eliminate the Goldstone boson Θ from the Lagrangian:

$$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(m + gC)^2 B_\mu B^\mu + \tfrac{1}{2}\partial_\mu C \, \partial^\mu C - \tfrac{1}{2}(m + \tfrac{1}{2}gC)^2 C^2$$

$$+ \frac{\mathrm{i}}{2} \bar{\lambda}\partial\!\!\!/\lambda + \frac{\mathrm{i}}{2} \bar{\chi}\partial\!\!\!/\chi - \frac{\mathrm{i}g}{2} \bar{\chi}\gamma^\mu \gamma_5 \chi B_\mu - (m + gC)\bar{\lambda}\gamma_5\chi. \tag{11.13}$$

The fermion mass term can be diagonalized by introducing field combinations

$$\lambda' = \frac{1}{\sqrt{2}}(\lambda + \gamma_5\chi); \qquad \chi' = \frac{1}{\sqrt{2}}(\lambda - \gamma_5\chi)$$

in which case the free part of the fermion Lagrangian is

$$\frac{i}{2}\bar{\lambda}'\slashed{\partial}\lambda' + \bar{\chi}'\slashed{\partial}\chi' - \frac{m}{2}(\bar{\lambda}'\lambda' + \bar{\chi}'\chi').$$

We see that the spectrum of the model contains an axial-vector B_μ, a scalar C and two Majorana spinors λ and χ, all with the same mass m. This mass degeneracy, together with the fact that the potential is zero at its minimum, is a strong indicator for unbroken supersymmetry.

In order to confirm this, we can study the following action [15] which involves a real general superfield V and its curl superfield W_α:

$$I = \left[\frac{1}{16} \int d^4x \, d^2\theta \, W^2 + \text{h.c.} \right] + \frac{m^2}{8g^2} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, (e^{-2gV} + 2gV). \tag{11.14}$$

This is clearly supersymmetric and can be expanded into components

$$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}\slashed{\partial}\lambda + \tfrac{1}{2}D^2 - \frac{m^2}{4g^2}[e^{-2gV}]_D - \frac{m^2}{2g}D.$$

The exponential term is tedious but straightforward to evaluate (use the Taylor expansion in θ and $\bar{\theta}$ rather than that for the exponential). The terms which involve the field components M and N are

$$-\frac{m^2}{4g^2}[e^{-2gV}]_D = \cdots + \frac{m^2}{2}e^{-2gC}\left[\left(M + \frac{g}{2}\bar{\chi}\gamma_5\chi\right)^2 + \left(N + \frac{g}{2}\bar{\chi}\chi\right)^2\right]$$

and give no contribution at all after the auxiliary fields M and N are eliminated. The part of the Lagrangian which involves the auxiliary field D gives

$$L = \cdots + \tfrac{1}{2}D^2 + \frac{m^2}{2g}(e^{-2gC} - 1)D = \cdots - \frac{m^4}{8g^2}(e^{-2gC} - 1)^2$$

and the total "on-shell" Lagrangian is found to be

$$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}\slashed{\partial}\lambda - \frac{m^4}{8g^2}(e^{-2gC} - 1)^2 - \frac{im^2g}{2}e^{-2gC}\bar{\chi}\gamma^\mu\gamma_5\chi A_\mu$$

$$+ m^2 e^{-2gC}[\tfrac{1}{2}\partial_\mu C\,\partial^\mu C + \tfrac{1}{2}i\bar{\chi}\slashed{\partial}\chi + \tfrac{1}{2}A_\mu A^\mu - \bar{\lambda}\chi] + \text{4-div}.$$

The field redefinitions

$$m\,e^{-gC}\chi \to \gamma_5\chi$$

$$e^{-gC} \to 1 + \frac{g}{m}\,C$$

$$A_\mu \to B_\mu$$

give exactly the Lagrangian (11.13) which has thus been shown to be supersymmetric.

The supersymmetric Higgs model can be seen either as a gauge model where gauge invariance but not supersymmetry is spontaneously broken, or as a real general multiplet in self-interaction. P. Fayet has observed [15] that the superfield action (11.13) for the latter picture can be derived from the action for the former one, eq. (11.6), by choosing not the Wess–Zumino gauge but what should be called the "Fayet gauge"

$$\phi = m/g \tag{11.15}$$

which is supersymmetric. The existence of such a gauge where apparently the matter fields have been gauged away is a consequence of supersymmetry – the degrees of freedom reappear in C and χ which now propagate and in A_μ which has now acquired a longitudinal part. The gauge choice (11.15) can be seen as the supersymmetric version of the procedure used in the Higgs model where a complex field is replaced by its modulus and phase, and the latter is then gauged to zero. Here, with a complex parameter superfield Λ, we can also gauge the modulus to a constant. Consequently, not only the phase reappears as the third degree of freedom of a massive vector, but also the other matter fields as a scalar and a Majorana spinor.

11.3. Broken supersymmetry

In the second case of fig. 11.1, where $\xi g > 0$, the minimum of the potential is at

$$\langle U\rangle = \tfrac{1}{2}\xi^2 \quad\text{and}\quad \langle A\rangle = 0. \tag{11.16}$$

No shift in fields is required, but after elimination of the auxiliary fields from the Lagrangian, we get the potential (11.9) which contains a mass term $\tfrac{1}{2}g\xi A^\dagger A$ for the scalars but no similar term for one of the spinors. Thus the mass degeneracy is lifted and supersymmetry is spontaneously broken – as expected for a model where the lowest energy is not zero but $\tfrac{1}{2}\xi^2$. A field which is not the θ-$\bar\theta$-independent part of a superfield, namely the auxiliary field D, has taken on a vacuum expectation value. This field appears in the transformation law for the gaugino field λ and consequently there is a field independent piece in the on-shell transformation law for λ:

$$\delta\lambda = \gamma_5\zeta\xi - i\sigma^{\mu\nu}\zeta\partial_\mu A_\nu + \frac{g}{2}\,\gamma_5\zeta A^\dagger A. \tag{11.17a}$$

This property, which perhaps is better written as

$$\{\lambda_\alpha, \bar{Q}^\beta\} = i(\gamma_5)_\alpha{}^\beta\,\xi + \cdots \tag{11.17b}$$

shows λ to be the *Goldstone fermion* ("goldstino") of the broken supersymmetry.

Note that since supersymmetry is not a local gauge invariance of the model, there is no "super-Higgs" effect and the goldstino is not "eaten up" to provide additional degrees of freedom for a massless gauge field. Such a thing can only happen in the context of supergravity where spontaneously broken local supersymmetry may occur.

11.4. The Fayet–Iliopoulos model

In the original paper by Fayet and Iliopoulos [13], super-QED was the theory whose supersymmetry was spontaneously broken, rather than the simpler gauge model discussed so far. Taking the super-QED Lagrangian, consisting of the L_{matter} of eq. (9.30) and the L_{Maxwell} of eq. (9.35) and adding to it the Fayet–Iliopoulos term (11.4), we get for the scalar part of the Lagrangian

$$L_{\text{pot}} = \tfrac{1}{2}D^2 - g(A_1B_2 - B_1A_2)D + \xi D + \sum_{i=1}^{2} (\tfrac{1}{2}F_i^2 + \tfrac{1}{2}G_i^2 - mA_iF_i - mB_iG_i) \tag{11.18}$$

which gives the usual potential $U = \tfrac{1}{2}\Sigma(\text{auxiliary fields})^2$ with equations of motion

$$D = g(A_1B_2 - B_1A_2) - \xi$$

$$F_i = mA_i; \qquad G_i = mB_i . \tag{11.19}$$

The condition for a supersymmetric vacuum, $\langle D \rangle = \langle F_i \rangle = \langle G_i \rangle = 0$, cannot be fulfilled for $m \neq 0 \neq \xi$, so that supersymmetry is always spontaneously broken for this model. Looking at the term quadratic in fields we find a non-diagonal scalar mass matrix which is diagonalized by the transformations

$$\tilde{A}_1 = \frac{1}{\sqrt{2}}(A_1 + B_2); \qquad \tilde{A}_2 = \frac{1}{\sqrt{2}}(A_2 - B_1)$$

$$\tilde{B}_1 = \frac{1}{\sqrt{2}}(A_1 - B_2); \qquad \tilde{B}_2 = \frac{1}{\sqrt{2}}(A_2 + B_1), \tag{11.20}$$

resulting in the potential

$$U = \tfrac{1}{2}(m^2 - \xi g)(\tilde{A}_1^2 + \tilde{A}_2^2) + \tfrac{1}{2}(m^2 + \xi g)(\tilde{B}_1^2 + \tilde{B}_2^2) + \tfrac{1}{8}g^2(\tilde{A}_1^2 + \tilde{A}_2^2 - \tilde{B}_1^2 - \tilde{B}_2^2)^2 + \tfrac{1}{2}\xi^2 . \tag{11.21}$$

Although we can assume that always $\xi g > 0$ (otherwise the roles of $\tilde{A}$ and $\tilde{B}$ are interchanged), we must distinguish two different cases:

(a) If $\xi g < m^2$, the minimum of the potential is at

$$\langle \tilde{A}_i \rangle = \langle \tilde{B}_i \rangle = 0 \qquad \text{and} \qquad \langle U \rangle = \tfrac{1}{2}\xi^2 . \tag{11.22}$$

In this case supersymmetry is spontaneously broken, but gauge invariance is not (the gauge transformations for the new fields are $\delta_g \tilde{A}_1 = \alpha \tilde{A}_2$; $\delta_g \tilde{A}_2 = -\alpha \tilde{A}_1$ and the same for $\tilde{B}_i$). The mass spectrum has become

$$m_{\tilde{A}_i} = \sqrt{m^2 - \xi g}\,; \qquad m_{\psi_i} = m\,; \qquad m_{\tilde{B}_i} = \sqrt{m^2 + \xi g}$$

$$m_{A_\mu} = m_\lambda = 0\,. \tag{11.23}$$

The mass splitting inside the matter multiplet establishes for sure that supersymmetry has been broken. The only massless spinor, λ, is the goldstino.

(b) If $\xi g > m^2$ the minimum of the potential is at a finite value for the matter fields. One choice is

$$\langle \tilde{A}_2 \rangle = \langle \tilde{B}_i \rangle = 0$$

$$\langle \tilde{A}_1 \rangle = \mu/g \qquad \text{with} \qquad \mu^2 \equiv 2(\xi g - m^2) \tag{11.24a}$$

and gauge invariance is spontaneously broken. Supersymmetry is as well, since

$$\langle U \rangle = \frac{m^2}{2g^2}(\mu^2 + m^2) = \tfrac{1}{2}\xi^2 - \frac{1}{2g^2}(\xi g - m^2)^2 \equiv \tfrac{1}{2}\xi^2 - \Delta U \tag{11.24b}$$

which is larger than zero but less than $\tfrac{1}{2}\xi^2$. A Higgs mechanism will eliminate the Goldstone scalar $\tilde{A}_2$, give the vector A_μ a mass and mix the three spinors of the model into mass eigenstates $\tilde{\lambda}$ and $\tilde{\psi}_i$. The details can be found in the original reference [13] and here it shall suffice to give the final mass spectrum:

$$m_{\tilde{\psi}_i} = \sqrt{m^2 + \mu^2}\,; \qquad m_{\tilde{\lambda}} = 0$$

$$m_{A_\mu} = m_{\tilde{A}_1} = \mu\,; \qquad m_{\tilde{B}_i} = \sqrt{2}\,m\,. \tag{11.25}$$

The goldstino is clearly that particular linear combination $\tilde{\lambda}$ of the spinors λ and ψ_i which has remained massless.

Figure 11.2 shows the behaviour of the potential for the two cases, drawn against $|\langle \tilde{A}_1 \rangle|$.

The spectrum of the Fayet–Iliopoulos model for various values of ξg can be summarised in fig. 11.3. A more detailed analysis will show that, in order not to break parity, the gauge field A_μ must be an axial vector. This is a consequence of the fact that the auxiliary field D which appears in the Fayet–Iliopoulos term would be a pseudoscalar were A_μ a vector. Thus the starting point of this analysis was not super-QED after all but rather a *chiral* $U(1)$ gauge theory.
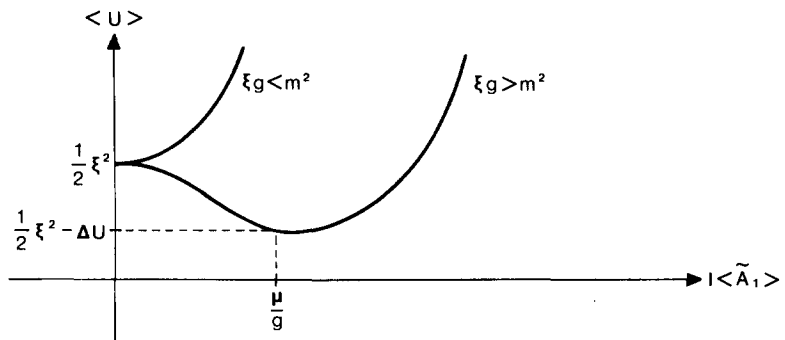
Fig. 11.2. The potential for the Fayet–Iliopoulos model.

Fig. 11.3. The mass spectrum of the Fayet–Iliopoulos model.

The *mass formula* (6.29) is fulfilled in the case of the Fayet–Iliopoulos model, but not for the minimal model with broken supersymmetry of the previous subsection.

1.5. Stability of mass spectra

A word is in order on the question of the behaviour of the mass spectrum under renormalisation. One of the most attractive features of supersymmetric field theories for model building is the stability of masses. This feature points a way around the "technical hierarchy problem" where a "spill-over" from a large mass scale tends to raise a small one and thus renders large mass gaps like the thirteen orders of magnitude between the GUT-mass and the W-mass very unnatural: the gap must be enforced order by order. This does not happen in those supersymmetric theories where mass terms do not get separately renormalised. The already mentioned non-renormalisation theorem for chiral superspace integrals provides such stability for mass terms like $\int d^4x\, d^2\theta\, m\phi^2$. In the present section, however, we have seen that the actual "physical" mass spectrum is determined not only by these terms but also by the value of the Fayet–Iliopoulos parameter ξ which multiplies an integral over the whole of superspace, $\int d^4x\, d^2\theta\, d^2\bar{\theta}\, \xi V$, and can thus a-priori receive independent quantum corrections. Answering the question of mass stability must therefore involve an investigation of the circumstances under which the Fayet–Iliopoulos term does indeed not get renormalised [22, 41], see also [31].

12. $N = 2$ supersymmetry

A large part of the excitement generated in theoretical physics by the concept of supersymmetry is due to the non-renormalisation theorems which hold for supersymmetric models. The property of quantum divergences to be less singular than one would ordinarily expect is very common and indeed present already in the simplest of all the models, the $N = 1$ scalar self-interaction models and the gauge models discussed so far. They are the ones used for the model building of super-GUTs.

The other exciting feature of supersymmetry, however, is the potential for true unification of the gravitational force with the other forces of nature. Although these forces are now understood to be the consequences of a "gauge principle", a true super-unification cannot be made apparent on the simple gauge models of $N = 1$ supersymmetry. First of all, an extension from Minkowskian supersymmetry to supergravity is clearly called for, and secondly, the gauge group must be much deeper ingrained in the structure of the theory than the "tacked on" gauge groups with which we have dealt so far. We saw at the beginning of section 9 that those gauge transformations which do *not* commute with supersymmetry are the ones which are intimately related to general coordinate transformations, and we anticipate that they will be the ones which in the end may provide super-unification ("gauged $N = 8$ supergravity" is the catch phrase in this context).

"Not commuting with supersymmetry" means more than just upsetting a particular non-supersymmetric gauge like the Wess–Zumino gauge. It means that the generators of the gauge symmetry do not commute with the generators of supersymmetry. We have seen in section 2 that the largest group whose generators do not commute with all the Q_{ai} ($i = 1, \ldots, N$) and all the $\bar{Q}_{\dot{a}}{}^i$ is U(N) or, in the presence of central charges, USp(N). Thus in order to ever have a non-Abelian gauge group in a gauged supergravity theory, the number N of independent spinorial symmetries must be larger than one.

This makes the study of *extended supersymmetry* imperative, and we shall begin this study outside of the context of supergravity, by looking at the simplest Minkowskian $N = 2$ models. Clearly that precious U(2) symmetry will only be a global invariance in these cases, since supersymmetry itself is ungauged. This point was discussed at the beginning of section 9.

12.1. A model with $N = 2$ supersymmetry

Since the helicities in a supermultiplet of particle states have at least a range of $\frac{1}{2}N$, the simplest $N = 2$ multiplet will contain spins $\frac{1}{2}$ and 0. This multiplet must, however, be either massless or have a non-trivial central charge, and we discard it for the time being (we shall later see that a self-interaction is not possible for it). The next simplest multiplet will already contain spin 1 and thus must be a *gauge multiplet* and massless. Its spectrum, taking into account the doubling necessary to meet the requirement of CPT-invariance, is

$$
\begin{array}{llccccc}
\underline{N=2} & \text{helicity:} & -1 & -\tfrac{1}{2} & 0 & \tfrac{1}{2} & 1 \\
& \text{states:} & 1 & 2 & 1+1 & 2 & 1
\end{array}
\qquad (12.1)
$$

and comparing this with the $N = 1$ case, we see that it is the superposition of a gauge multiplet with a chiral multiplet:

helicity:	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	
$\underline{\underline{N=1}}$ gauge multiplet:	1	1		1	1	(12.2)
chiral multiplet:		1	$1+1$	1		.

As a first try, we therefore look at the Lagrangian for the non-Abelian gauge theory of a single massless chiral multiplet in the adjoint representation of the gauge group. This is the sum of the Lagrangians (10.45) and (10.51). We eliminate the auxiliary fields and define

$$\lambda_1 \equiv \lambda ; \qquad \lambda_2 \equiv \psi . \tag{12.3}$$

The resulting Lagrangian (to avoid future confusion, the fields A and B of the chiral multiplet are renamed as M and N)

$$L = \text{tr}[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}i\bar{\lambda}_i\gamma^\mu \, \nabla_\mu\lambda_i + \tfrac{1}{2}\nabla_\mu M \, \nabla^\mu M + \tfrac{1}{2}\nabla_\mu N \, \nabla^\mu N$$

$$- i\bar{\lambda}_2[\lambda_1, M] - i\bar{\lambda}_2\gamma_5[\lambda_1, N] + \tfrac{1}{2}[M, N]^2] \tag{12.4}$$

has an O(2) symmetry which is more apparent if we rewrite

$$\bar{\lambda}_2^a \lambda_1^b c_{abc} = -\tfrac{1}{2}\varepsilon_{ij}\bar{\lambda}_i^a \lambda_j^b c_{abc} \qquad \text{(same for } \bar{\lambda}\gamma_5\lambda\text{)} \tag{12.5}$$

(this equation follows from the antisymmetry of the gauge group's structure constants c_{abc} and from eqs. (A.45) in the Appendix). Under this O(2) the gauge field A_μ is a singlet, but the old gaugino λ is part of a doublet. Since under the old $N=1$ supersymmetry transformations $\delta A_\mu = i\bar{\zeta}\gamma_\mu\lambda$, supersymmetry cannot commute with the O(2). Its generators must be part of a doublet, the other supersymmetry transformation being $\delta' A_\mu = i\bar{\zeta}'\gamma_\mu\psi$. A more detailed analysis will indeed show that the Lagrangian (12.4) is a density under the following set of transformations with two Majorana parameters λ_1 and λ_2:

$$\delta A_\mu = i\bar{\zeta}_i\gamma_\mu\lambda_i$$

$$\delta M = \varepsilon_{ij}\bar{\zeta}_i\lambda_j$$

$$\delta N = \varepsilon_{ij}\bar{\zeta}_i\gamma_5\lambda_j \tag{12.6}$$

$$\delta\lambda_i = -\tfrac{1}{2}i\sigma^{\mu\nu}\zeta_i F_{\mu\nu} + i\varepsilon_{ij}\gamma^\mu\nabla_\mu(M + \gamma_5 N)\zeta_j - i\gamma_5\zeta_i[M, N]$$

which lead to the algebra:

$$[\delta^{(1)}, \delta^{(2)}] = 2i\bar{\zeta}_i^{(1)}\gamma^\mu\zeta_i^{(2)}\partial_\mu + \delta_{\text{gauge}} + \text{field equations} . \tag{12.7}$$

Ignoring the gauge transformation (it arises because we are in a Wess–Zumino gauge) and the field equations (they arise because we are in an on-shell formulation without auxiliary fields), we see the relationship with the centre piece

$$\{Q_i, \bar{Q}_j\} = 2\delta_{ij}\gamma^\mu P_\mu \qquad (12.8)$$

of the $N = 2$ superalgebra.

The motivation for studying extended supersymmetry was its non-Abelian internal symmetry group. In the present example, however, it seems that the internal symmetry is actually only an O(2), not the U(2) we had hoped for. Remembering the remarks made after eq. (4.3), this is not surprising: as long as we use Majorana spinors of the form (4.1), our notation will hide any possible symmetry larger than O(N). One way to overcome this problem is to revert from Majorana spinors to chiral notation with 2-component spinors. However, as said before, presentation of results is much neater when using 4-component spinors. Fortunately, there is a possibility of constructing an SU(2)-covariant 4-component spinor from the 2-component spinors $\lambda_{\alpha i}$ and $\bar{\lambda}_{\dot\alpha}^i \equiv (\lambda_{\alpha i})^\dagger$: we define a *symplectic Majorana spinor* λ^i by

$$\lambda^i \equiv \begin{bmatrix} -i\varepsilon^{ij}\lambda_{\alpha j} \\ \bar{\lambda}^{\dot\alpha i} \end{bmatrix}; \qquad \bar{\lambda}_i = (\lambda_i^\alpha, i\varepsilon_{ij}\bar{\lambda}^j{}_{\dot\alpha}) \qquad (12.9)$$

which satisfies a symplectic reality condition [56]

$$\lambda^i = \varepsilon^{ij}\gamma_5 C \bar{\lambda}_j^T . \qquad (12.10)$$

The word "symplectic" refers to the antisymmetric tensor ε_{ij} which is used as a metric to raise and lower SU(2) indices. In general, reality conditions of this type are only covariant under symplectic groups, like SU(2) = Sp(2), just as ordinary Majorana conditions are only covariant under orthogonal groups.

In this SU(2) covariant notation, the Lagrangian (12.4) becomes

$$L = \mathrm{tr}[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}i\bar{\lambda}_i\gamma^\mu \nabla_\mu\lambda^i + \tfrac{1}{2}\nabla_\mu M \nabla^\mu M + \tfrac{1}{2}\nabla_\mu N \nabla^\mu N$$

$$- \tfrac{1}{2}\bar{\lambda}_i[\lambda^i, M] - \tfrac{1}{2}\bar{\lambda}_i\gamma_5[\lambda^i, N] + \tfrac{1}{2}[M, N]^2] \qquad (12.11a)$$

which is clearly invariant under the SU(2) which rotates the spinor doublet, but also under the chiral U(1) transformations

$$M + iN \to e^{-2i\alpha}(M + iN)$$

$$\lambda^i \qquad \to e^{\alpha\gamma_5}\lambda^i \qquad (12.12)$$

$$A_\mu \qquad \to A_\mu .$$

The transformation laws (12.6) are in this notation:

$$\delta A_\mu = i\bar{\zeta}_i\gamma_\mu\lambda^i$$

$$\delta M = i\bar{\zeta}_i\lambda^i$$

$$\delta N = i\bar{\zeta}_i\gamma_5\lambda^i \qquad (12.13a)$$

$$\delta\lambda^i = -\tfrac{1}{2}i\sigma^{\mu\nu}\zeta^i F_{\mu\nu} - \gamma^\mu \nabla_\mu(M + \gamma_5 N)\zeta^i - i\gamma_5\zeta^i[M, N] .$$

An off-shell version of these transformation laws is known [29]. It involves a real SU(2) triplet of auxiliary fields D with U(1) weight 0. For the off-shell version, the transformation laws (12.13a) must be augmented by

$$\delta\lambda^i = \cdots - i\zeta^j\tau_j{}^i D$$
$$\delta D = \tau_i{}^j \bar{\zeta}_j\gamma^\mu \nabla_\mu\lambda^i + i\tau_i{}^j \bar{\zeta}_j[\lambda^i, M] + i\tau_i{}^j\bar{\zeta}_j\gamma_5[\lambda^i, N] \tag{12.13b}$$

and the Lagrangian (12.11) picks up the additional term

$$L = \cdots + \tfrac{1}{2}\mathrm{tr}\, D^2 . \tag{12.11b}$$

The inclusion of D will drop the term "+ field equations" from the algebra (12.7), but the term "+ δ_gauge" is still there:

$$[\delta^{(1)}, \delta^{(2)}] = 2i\bar{\zeta}_i^{(1)}\gamma^\mu\zeta^{i(2)}\partial_\mu + \delta_\mathrm{gauge} , \tag{12.14a}$$

with the following field dependent parameter for the gauge transformation

$$\alpha = -2i\bar{\zeta}_i^{(1)}\gamma^\mu\zeta^{i(2)}A_\mu + 2i\bar{\zeta}_i^{(1)}(M + \gamma_5 N)\zeta^{i(2)} . \tag{12.14b}$$

12.2. The N = 2 matter multiplet ("hypermultiplet")

When a French super-marché carries not only food and drink but also car spares, garden furniture and ladies' underwear, it becomes an hyper-marché. Correspondingly, P. Fayet called $N = 2$ super-symmetry "hyper-symmetry" [14]. Whereas that name has not stuck in general, the matter multiplet of $N = 2$ supersymmetry is still called the "hypermultiplet". The well chosen use of that word is likely to produce a glimmer in the eyes of the true cognoscenti.*

If $N > 1$, any massive multiplet will contain particles of spin ≥ 1, unless there is a central charge Z which satisfies the condition [57, 60, 76]

$$Z^2 = P_\mu P^\mu , \tag{12.15}$$

so that the eigenvalue Z of the central charge equals plus or minus the mass, see subsection 3.4. The spectrum of the hypermultiplet is given by eq. (3.40). So far no explanation has been given for the mysterious doubling of states in that spectrum. Without that doubling, the spectrum would be the same as for the $N = 1$ chiral multiplet, and the question therefore arises whether we can represent $N = 2$ supersymmetry on a single chiral multiplet of $N = 1$. A second supersymmetry generator S_α, different from Q_α, would have to see the multiplet as *anti*-chiral, or S_α would just be a multiple of Q_α again. The ansatz

$$[A, \bar{Q}] = 0; \qquad [A, S] = 0$$
$$[A, Q] = 2i\omega; \qquad [A, \bar{S}] = \alpha\bar{\psi}$$

---

* Early references to $N = 2$ models are [11] and, particularly, ref. [57] where the matter multiplet first appeared in print.

is already the most general one. At the next level, the required properties $\{Q, \bar{Q}\} = \{S, \bar{S}\} = 2\sigma^\mu P_\mu$ lead to $\{\psi, \bar{Q}\} = \partial\!\!\!/A$ and $\{S, \bar{\psi}\} = 2i\alpha^{-1}\partial\!\!\!/A$, and by Hermitian conjugation to

$$\{Q, \bar{\psi}\} = \partial\!\!\!/A^\dagger; \qquad \{\psi, \bar{S}\} = -\frac{2i}{\alpha^*}\partial\!\!\!/A^\dagger.$$

Thus

$$[A, \{Q, \bar{S}\}] = 2i\{\psi, \bar{S}\} + \alpha\{Q, \bar{\psi}\} = (4/\alpha^* + \alpha)\partial\!\!\!/A^\dagger,$$

an expression which must vanish because we require that $\{Q, \bar{S}\} = 0$. This implies the trivial solution $A = \text{const}$.

A different ansatz is therefore called for which involves *two* complex fields A_i. On them two supersymmetry generators Q_i can act. Since we only want to double the spectrum of the chiral multiplet, the commutator $[A_i, Q_{\alpha j}]$ should be expressible in terms of no more than two different chiral spinors. Since the most general ansatz for that commutator would be $\varepsilon_{ij}\psi_\alpha + \psi_{\alpha(ij)}$, we see that the triplet $\psi_{\alpha(ij)}$ must be ruled out. The hypermultiplet must therefore satisfy a *constraint* which reads [60]

$$[A_i, \bar{Q}_j] + [A_j, \bar{Q}_i] = 0 \tag{12.16}$$

in a notation where Q^i is a symplectic Majorana spinor. In an $N = 2$ superspace language this constraint would read

$$D_{\alpha(i}\phi_{j)} = \bar{D}_{\dot\alpha(i}\phi_{j)} = 0.$$

It defines the hypermultiplet just as the chiral constraint (3.43) defined the chiral multiplet. Indeed, just as for the chiral multiplet, we can go through a procedure like the "seven easy steps" of subsection 3.6 and construct the entire multiplet from this constraint and knowledge of the superalgebra. An attempt to do this without a central charge will, however, lead to free massless equations of motion for the fields. Only if we allow for a *central charge*, do we get an acceptable result: a multiplet

$$\phi_i = (A_i; \psi; F_i) \tag{12.17}$$

of two complex scalars $A_i(x)$, a Dirac spinor $\psi_\alpha(x)$ and two complex auxiliary scalars $F_i(x)$, with supersymmetry transformations as follows:

$$\delta A_i = 2\bar{\zeta}_i\psi$$
$$\delta\psi = -i\zeta^i F_i - i\partial\!\!\!/\zeta^i A_i \tag{12.18}$$
$$\delta F_i = 2\bar{\zeta}_i\partial\!\!\!/\psi.$$

These transformations satisfy the algebra

$$[\delta^{(1)}, \delta^{(2)}] = 2i\bar{\zeta}_i^{(1)}\gamma^\mu\zeta^{i(2)}\partial_\mu + 2i\bar{\zeta}_i^{(1)}\zeta^{i(2)}\delta_z \tag{12.19}$$

with the action of the central charge, δ_z, given by

$$\delta_z A_i = F_i$$

$$\delta_z \psi = \not{\partial} \psi$$

$$\delta_z F_i = \Box A_i .$$
(12.20)

It is easy to check that the central charge commutes with supersymmetry,

$$[\delta, \delta_z] = 0$$
(12.21)

and that

$$\delta_z^2 = \Box ,$$
(12.22)

which is just the condition (12.15) which ensures that there are no higher spins in the multiplet than $\frac{1}{2}$.

In order to construct an invariant from the hypermultiplet, we use the technique of finding a contragradient multiplet, similar to subsection 4.6. The hypermultiplet turns out to be contragradient to its own Hermitian adjoint and we find for any two hypermultiplets ϕ_i and

$$\phi_i' = (a_i; \chi; f_i)$$

that the product

$$(\bar{\phi}'^i \cdot \phi_i) = i a^{\dagger i} F_i - i f^{\dagger i} A_i + 2\bar{\chi}\psi$$
(12.23)

is a density. This product does not involve derivatives and can only be used for mass terms in the Lagrangian; it is the $N = 2$ generalisation of $(\phi \cdot \phi)_F$. Since, however, $\partial_z \phi_i$ is also a hypermultiplet, with components which do involve derivatives,

$$\delta_z \phi_i = (F_i; \not{\partial}\psi; \Box A_i)$$
(12.24)

(it plays the role of the kinetic multiplet), it is easy to construct a *free Lagrangian* from (12.23):

$$L = \frac{i}{2}(\bar{\phi}^i \cdot \delta_z \phi_i) + \frac{m}{2}(\bar{\phi}^i \cdot \phi_i)$$

$$= \frac{1}{2}\partial_\mu A^{\dagger i}\,\partial^\mu A_i + \frac{1}{2}F^{\dagger i}F_i + i\bar{\psi}\not{\partial}\psi + m[\frac{1}{2}iA^{\dagger i}F_i - \frac{1}{2}iF^{\dagger i}A_i + \bar{\psi}\psi] + 4\text{-div} .$$
(12.25)

The equations of motion are

$$\delta_z \phi_i = im\phi_i$$
(12.26)

when written in multiplet form. In components, these are the equations for free fields with mass m once the auxiliary fields F_i have been eliminated.

Note that the non-compact Z-transformations (12.20) become compact phase transformations on-shell, with a charge that is equal to the mass – just the multiplet shortening condition! In the absence of a mass term, the central charge disappears entirely on-shell, as it should according to subsection 3.4.

12.3. The hypermultiplet in interaction

The Lagrangian (12.25) describes free fields. In the case of $N = 1$, it was possible to write down a trilinear self-interaction term for the chiral multiplet. This resulted in the Wess–Zumino model which was a supersymmetric generalisation of the self-interacting scalar $\lambda\phi^4$ model. A similar construction is not possible for the hypermultiplet. This is quite obvious if we insist on SU(2) invariance of the Lagrangian since half-integer spins go together with integer "iso"-spins, and thus there is no third-order invariant. But even if SU(2) invariance is sacrificed, no supersymmetric renormalisable self-coupling terms exist [47]. We can therefore say that

the only interactions of $N = 2$ matter are gauge interactions.

Ideally, we should like to have a full superspace treatment of the theory. Unfortunately, however, no such treatment exists for the non-Abelian case (there is a solution of the superspace constraints for the Abelian case [46], and the full impact of the recent development of "harmonic superspace" needs to be awaited [23]). For the time being, we are forced to proceed in a rather pedestrian way.

The problem we face is that the gauge multiplet is already in a Wess–Zumino gauge, resulting in the additional δ_{gauge} in the algebra (12.14). If the gauge multiplet is to interact with matter in a supersymmetric fashion, the same algebra must be represented on all fields involved. This means that we must extend the algebra of the transformations of the gauge fields to include the central charge, and the algebra on the matter fields to include the gauge transformations.

The former is easily done, we just assume the fields of the gauge multiplet to be invariant under Z-transformations:

$$\delta_z A_\mu = \delta_z M = \delta_z N = \delta_z \lambda^i = \delta_z D = 0 .$$
(12.27)

The algebra is now that of eq. (12.14) with the additional δ_z-term of (12.19). Its generic structure is

$$[\delta^{(1)}, \delta^{(2)}] = \delta_{\text{translation}} + \delta_{\text{central charge}} + \delta_{\text{gauge}} ,$$
(12.28a)

and on a matter multiplet it looks like this:

$$[\delta^{(1)}, \delta^{(2)}] = 2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\mu\zeta^{i(2)}\nabla_\mu + 2\mathrm{i}\bar{\zeta}_i^{(1)}(\delta_z - \mathrm{i}M - \mathrm{i}\gamma_5 N)\zeta^{i(2)} .$$
(12.28b)

The fields M and N are Lie-algebra valued and act on the gauge group representation indices of the fields of the matter multiplet. We see that a non-zero vacuum expectation value for M or N will induce a central charge [76].

We can impose this algebra on a doublet A_i of complex scalar fields, just as we could impose the simpler algebra without gauge transformations. The constraint (12.16) is to remain unchanged. Then the result of a rather lengthy calculation will be the following modified transformation laws for the fields of the *gauge hypermultiplet,*

$$\delta A_i = 2\bar{\zeta}_i\psi$$

$$\delta\psi = -\mathrm{i}\zeta^i F_i - (\mathrm{i}\gamma^\mu\nabla_\mu + M + \gamma_5 N)\zeta^i A_i$$
(12.29)

$$\delta F_i = 2\bar{\zeta}_i(\gamma^\mu\nabla_\mu + \mathrm{i}M - \mathrm{i}\gamma_5 N)\psi - 2\bar{\zeta}_i\lambda^j A_j ,$$

together with modified central charge transformations

$$\delta_z A_i = F_i$$

$$\delta_z \psi = (\chi^\mu \nabla_\mu + iM - i\gamma_5 N)\psi - \lambda^i A_i$$

$$\delta_z F_i = (\nabla^\mu \nabla_\mu + M^2 + N^2)A_i - 2i\bar\lambda_i\psi - \tau_i{}^j DA_j + 2iMF_i .$$

(12.30)

The product density (12.23) picks up another term and becomes

$$(\bar\phi'^i \cdot \phi_i) \equiv ia^{\dagger i}F_i - if^{\dagger i}A_i + 2\bar\chi\psi + 2a^{\dagger i}MA_i .$$

(12.31)

The central charge transformations still commute with supersymmetry, therefore $\delta_z\phi_i$ is still the kinetic multiplet, and we can write our Lagrangian as

$$L = \frac{i}{2}(\bar\phi^i \cdot \delta_z\phi_i) + \frac{m}{2}(\bar\phi^i \cdot \phi_i)$$

$$= \tfrac{1}{2}\nabla_\mu A^{\dagger i}\nabla^\mu A_i + \tfrac{1}{2}F^{\dagger i}F_i + i\bar\psi\gamma^\mu \nabla_\mu\psi + iA^{\dagger i}\bar\lambda_i\psi - i\bar\psi\lambda^i A_i - \bar\psi(M - \gamma_5 N)\psi$$

$$- \tfrac{1}{2}A^{\dagger i}(M^2 + N^2)A_i + \tfrac{1}{2}A^{\dagger i}\tau_i{}^j DA_j + m(\tfrac{1}{2}iA^{\dagger i}F_i - \tfrac{1}{2}iF^{\dagger i}A_i + \bar\psi\psi) + mA^{\dagger i}MA_i + \text{4-div}.$$

(12.32)

The equations of motion remain $\delta_z\phi_i = im\phi_i$ but the left-hand side is now given by the more complicated eqs. (12.30) with all their interaction terms.

12.4. Chiral fermions?

We have already seen how the $N = 2$ gauge multiplet (12.13) originated from a gauge multiplet and a chiral multiplet of $N = 1$. Similarly, we now look at the transformation laws (12.18) for a free hypermultiplet and try to analyse its content in terms of $N = 1$ multiplets. Of the two chiral spinors $\zeta_{\alpha i}$ from which the symplectic Majorana parameter ζ^i is built, we take $\zeta_{\alpha i}$, call it ζ_α, break the Dirac spinor of the hypermultiplet into chiral parts ψ_α and $\bar\chi_{\dot\alpha}$, and find the following transformation laws (in chiral notation)

$$\delta A_1 = 2\zeta\psi; \qquad \delta A_2 = -2i\bar\zeta\bar\chi$$

$$\delta\psi = \zeta F_2 - i\partial\!\!\!/\bar\zeta A_1; \qquad \delta\bar\chi = -i\bar\zeta F_1 + \partial\!\!\!/\zeta A_2$$

(12.33)

$$\delta F_2 = -2i\bar\zeta\partial\!\!\!/\psi; \qquad \delta F_1 = 2\zeta\partial\!\!\!/\bar\chi .$$

By comparison with eqs. (3.50) and (3.53), these can be recognised as those of a chiral multiplet $(A_1; \psi; -F_2)$ and an anti-chiral multiplet $(iA_2; \bar\chi; iF_1)$. This was to be expected since the hypermultiplet contains a full Dirac spinor whose right-handed and left-handed components must belong to a chiral and anti-chiral multiplet, respectively. This has the important consequence that a matter multiplet of $N = 2$ supersymmetry which transforms under some representation of a gauge group will always contain both a right-handed and a left-handed fermion in that representation, and therefore with the same internal quantum numbers. Thus even if the mass is zero, the representations of the gauge group are necessarily vector-like. The same is, of course, true for all higher-N supersymmetric models, since their

symmetry groups always contain $N = 2$ supersymmetry as a subsymmetry. Realistic models therefore require the breakdown of all but the last of the supersymmetries already at very high energies in order to obtain the V-A structure of the weak interactions.

12.5. $N = 2$ supersymmetry from an $N = 1$ model

After the considerations in subsections 12.1 and 12.4, we now know that the most general $N = 2$ gauge theory, described by the Lagrangians (12.11) and (12.32), can be seen in the light of $N = 1$ supersymmetry as a model with three matter multiplets. Two of these are possibly massive and in representations of the gauge group which are conjugate to each other, the third is massless and in the adjoint representation.

A different question may now be asked: when does an $N = 1$ supersymmetric gauge theory, described in general by the Lagrangian (10.53), have a second supersymmetry? This requires a second gaugino, so there must be *one matter multiplet* Ψ in the adjoint representation. Secondly, the theory must be *vector-like*, so if there are further chiral matter multiplets ϕ_a in some representation of the gauge group (not necessarily irreducible) then there must be corresponding multiplets ϕ^a in the conjugate representation ($\bar{\phi}^a \equiv (\phi_a)^\dagger$ won't do). The most general $N = 1$ gauge theory for these (compare subsection 10.10) is described by a superpotential*

$$V = b^a \, \phi_a + b'_a \, \phi^a + b_a{}^b \, \Psi_b{}^a - m\phi^a \, \phi_a + g\phi^a \Psi_a{}^b \, \phi_b \,, \tag{12.34}$$

or rather a slight generalisation of it: the parameters m and g can actually take different values for different irreducible parts of ϕ_a. Also, if the same irreducible representation appears more than once, the mass term and the third-order coupling term may not be simultaneously diagonal.

This $N = 1$ theory is the most general one with the right kind of spectrum, and we can now connect it to the $N = 2$ theory (12.32). After eliminating the auxiliary fields, the spinor from Ψ is combined with the gaugino λ to form a gaugino doublet λ^i, the spinor from ϕ_a is combined with that from $(\phi^a)^\dagger$ to form the Dirac spinor of the hypermultiplet, the scalars from ϕ_a and ϕ^a are combined into the complex doublet A_i, and finally the scalars from Ψ are renamed M and N. After a lengthy calculation, it emerges that this particular $N = 1$ model can be recast into the form of the $N = 2$ Lagrangian (12.32) (with auxiliary fields eliminated), *provided* that indeed the mass and third-order coupling terms are simultaneously block-diagonal and that

$$b^a = b'_a = b_a{}^b = 0 \tag{12.35}$$

$$|g| = 1 \,. \tag{12.36}$$

The latter equation is most important: the essential difference between the $N = 1$ and the $N = 2$ theories is that the former still has two coupling parameters, g and the gauge-coupling constant which was set $= 1$ in both (10.53) and (12.32). The phase of g can be absorbed into redefinitions of fields, and the correct way of stating the condition $|g| = 1$ for $N - 2$ supersymmetry is that

---

* If ϕ_a is in a real representation, there may be more terms, e.g. $m'\phi_a\phi_a$ or $d^{abc}\phi_a\phi_b\phi_c$; $N = 2$ supersymmetry will rule these out.

the super-ϕ^3 coupling constant must be equal to the gauge-coupling constant for each irreducible pair of matter multiplets ϕ_a and ϕ^a.

It is interesting to note that the higher symmetry is apparent only in the on-shell version of the Lagrangian. The sets of auxiliary fields for $N = 1$ and $N = 2$ are quite different and cannot be translated into each other.

The constants b are related to possible F_i and D terms in the $N = 2$ Lagrangian which would explicitly break SU(2), but not supersymmetry nor gauge invariance for an Abelian gauge group. The question of such $N = 2$ Fayet–Iliopoulos terms and of the spontaneous symmetry breaking which they may induce goes beyond the scope of this article, except for one point to be made: if one supersymmetry is spontaneously broken, then both are. This is a consequence of eq. (2.29) which says that the Hamiltonian can be expressed as sum of the squares of each of the supersymmetries separately. Thus the condition $\langle E \rangle \neq 0$ for the spontaneous breaking of one supersymmetry is sufficient to ensure the breaking of the other one as well. This situation is different for supergravity, and we arrive at interesting conclusions: Nature can have extended supersymmetry only at some high energy because of the observed chiral lepton spectrum. Yet one supersymmetry should survive to relatively low energies in order to stabilise the GUT hierarchy mass gap. Spontaneous breaking cannot reduce $N \geq 2$ to $N = 1$ supersymmetry in a Poincaré invariant setting, and we are left to conclude that at least the high-energy breaking from extended to simple supersymmetry must be somehow related to gravity effects.

12.6. $SU(2) \times SU(2)$ invariance and the real form of the hypermultiplet [4]

If the hypermultiplet is in a *real representation* of the gauge group, then the superpotential for the corresponding $N = 1$ theory can be written as

$$V = -m\phi_a^1 \phi_a^2 + \phi_a^1 \Psi_{ab} \phi_b^2 \tag{12.37}$$

with ϕ_a^1 and ϕ_a^2 the two chiral multiplets contained in the hypermultiplet. For a real representation we have $\Psi_{ab} = -\Psi_{ba}$, and in the *absence of the mass term* the superpotential can be cast into an SU(2)-invariant form

$$V = \tfrac{1}{2}\varepsilon_{i'j'} \phi_a^{i'} \Psi_{ab} \phi_b^{j'} . \tag{12.38}$$

Since whole multiplets carry the indices of this additional global symmetry group, the new SU(2) will commute with supersymmetry. It is therefore distinct from the old SU(2) which, e.g., rotated the A_i as a doublet. We distinguish the two groups by use of primed and unprimed indices.

Let us now explicitly establish the SU(2) × SU(2) invariance of the model. The components of A_i and F_i will be combined into 2 × 2 matrices $A_i^{i'}$ and $F_i^{i'}$ with reality conditions of the form

$$(A_i^{i'})^\dagger = -\varepsilon^{ij} \varepsilon_{i'j'} A_j^{j'} \tag{12.39}$$

(same for F). From the Dirac spinor ψ a symplectic Majorana doublet $\psi^{i'}$ will be formed with reality condition

$$\psi^{i'} = \varepsilon^{i'j'} \gamma_5 C \bar{\psi}_{j'}^{\mathrm{T}} . \tag{12.40}$$

The explicit expressions are

$$A_i^{1'} = A_i; \qquad A_i^{2'} = \varepsilon_{ij}A^{\dagger j}$$

$$\psi^{1'} = \psi; \qquad \psi^{2'} = -\gamma_5\psi^c, \tag{12.41}$$

and we see that this is only possible if A_i and $A^{\dagger i}$ are in the same representation of the gauge group (the representation must therefore be real). A pseudo-real representation won't do: if we write

$$A_i^{2'} = \omega\varepsilon_{ij}A^{\dagger j}$$

with ω some metric for the representation of the gauge group, then

$$(A_i^{1'})^\dagger = A^{\dagger i} = -\omega^{-1}\varepsilon_{12}\varepsilon^{ij}A_j^2$$

$$(A_i^{2'})^\dagger = \omega^*\varepsilon^{ij}A_j = -\omega^*\varepsilon_{21}\varepsilon^{ij}A_j^1, \tag{12.42}$$

and we can have an $SU(2) \times SU(2)$ covariant reality condition for $A_i^{i'}$ only if $\omega^* = \omega^{-1}$. This is possible for orthogonal but not for symplectic metrics. Interestingly enough, *pseudo-real representations* have a different property: they allow symplectic reality conditions like (12.39–40) which involve the gauge group indices rather than i' and j'. Consequently, such gauge theories need to have only half as many matter fields as one would naively expect [47]. An $SU(2)$-gauge theory can, e.g., be formulated on a single hypermultiplet rather than a doublet of them (this amounts to gauging the "new" $SU(2)$).

We can rewrite the entire matter part (12.32) of the Lagrangian in terms of the new quantities:

$$L = -\tfrac{1}{4}\varepsilon^{ij}\varepsilon_{i'j'}(\nabla_\mu A_i^{i'} \nabla^\mu A_j^{j'} + F_i^{i'} F_j^{j'} - A_i^{i'}(M^2 + N^2) A_j^{j'} + \tau_j^k A_i^{i'} DA_k^{j'})$$

$$+ \tfrac{1}{2}i\bar{\psi}_{i'} \gamma^\mu \nabla_\mu\psi^{i'} - i\bar{\psi}_{i'} \lambda^i A_i^{i'} - \tfrac{1}{2}\bar{\psi}_{i'} (M - \gamma_5 N)\psi^{i'}, \tag{12.43}$$

and the $SU(2) \times SU(2)$ invariance becomes manifest. The transformation laws (12.29–30) remain unchanged, except that an index i' is tacked onto all matter fields. That this is consistent with the definitions (12.41) is a nice check on algebraic correctness!

The perhaps somewhat awkward reality condition (12.39) can be solved in terms of four real fields A and $\mathbf{A}$ by

$$A_i^{i'} = i\delta_i^{i'} A + \tau_i^{i'} \mathbf{A} \tag{12.44}$$

(same for F). This leads to a *real form of the hypermultiplet* where the two $SU(2)$ groups have been identified and only the diagonal $SU(2)$ subgroup is still manifest. In this notation, the Lagrangian is

$$L = \tfrac{1}{2}\nabla_\mu A\nabla^\mu A + \tfrac{1}{2}\nabla_\mu \mathbf{A} \nabla^\mu \mathbf{A} + \tfrac{1}{2}i\bar{\psi}\gamma^\mu \nabla_\mu\psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}\mathbf{F}^2$$

$$+ \bar{\psi}\lambda A - i\bar{\psi}\tau^T\lambda\mathbf{A} - \tfrac{1}{2}\bar{\psi}(M - \gamma_5 N)\psi$$

$$- \tfrac{1}{2}A(M^2 + N^2)A - \tfrac{1}{2}\mathbf{A}(M^2 + N^2)\mathbf{A} - iADA - \tfrac{1}{2}i\mathbf{A} \cdot (\mathbf{D} \times \mathbf{A}), \tag{12.45}$$

with suppressed $SU(2)$ spinor indices on ψ and λ.

13. $N = 4$ supersymmetry

The largest amount of supersymmetry that can be represented on a particle multiplet with spins ≤ 1, is $N = 4$. Higher spins mean non-renormalisable couplings, and thus we reach, with $N = 4$, the limit of supersymmetric flat-space theory. Correspondingly, the $N = 4$ model [27] is called *maximally extended*. Any multiplet (even massless ones) will contain spin-1 particles, and thus all $N = 4$ models must be constructed solely from gauge multiplets of $N = 4$. This makes the particles necessarily massless and there will be no central charges on-shell, cf. the discussion after eq. (3.23).

13.1. The $N = 4$ super-Yang-Mills theory

The spectrum of the only $N = 4$ multiplet without higher spins is given by (3.12). This is a gauge multiplet because it contains spin-1, there is no matter multiplet and we conclude that the only interaction can be that of a non-Abelian gauge field A_μ in interaction with itself and its superpartners in the multiplet, the four gauginos and six scalars, all in the adjoint representation of the gauge group.

An $N = 4$ model will by default also be an $N = 1$ and an $N = 2$ model. We already know the most general $N = 1$ gauge theory with the required spectrum, it is described by a gauge multiplet and three massless chiral multiplets, all in the adjoint representation of the gauge group. The requirement of $N = 2$ supersymmetry (see subsection 12.5) leaves only one possible superpotential for these (up to a phase – we choose a convenient one):

$$V = \text{i tr } \phi_1[\phi_2, \phi_3] \, . \tag{13.1}$$

There is no free parameter whatsoever left to play with and the model either has the required $N = 4$ supersymmetry or it doesn't. Fortunately, it does.

To show this, we start with the Lagrangian for the $N = 1$ theory:

$$L = \text{tr}\Big(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}i\bar{\lambda}\gamma^\mu \, \nabla_\mu\lambda + \tfrac{1}{2}D^2$$

$$+ \sum_{i=1}^{3} \Big(\tfrac{1}{2}\nabla_\mu A_i\nabla^\mu A_i + \tfrac{1}{2}\nabla_\mu B_i \, \nabla^\mu B_i + \frac{i}{2}\,\bar{\psi}_i\gamma^\mu \, \nabla_\mu\psi_i + \tfrac{1}{2}F_i^2 + \tfrac{1}{2}G_i^2$$

$$- \text{i}[A_i, B_i]D - \text{i}\bar{\psi}_i[\lambda, A_i] - \text{i}\bar{\psi}_i\gamma_5[\lambda, B_i] \Big)$$

$$- \frac{i}{2}\sum_{ijk} \varepsilon_{ijk}(\bar{\psi}_i[\psi_j, A_k] - \bar{\psi}_i\gamma_5[\psi_j, B_k] + [A_i, A_j]F_k - [B_i, B_j]F_k + 2[A_i, B_j]G_k)\Big) \, . \tag{13.2}$$

We now establish an O(4) symmetry which does not commute with supersymmetry. We define four Majorana spinors by

$$\lambda_i = \psi_i \qquad \text{for } i = 1, 2, 3$$

$$\lambda_4 = \lambda \, , \tag{13.3a}$$

and antisymmetric 4×4 matrices of scalars and pseudoscalars by

$$A_{ij} = -\varepsilon_{ijk}A_k ; \qquad B_{ij} = \varepsilon_{ijk}B_k$$

$$A_{4i} = -A_{i4} = A_i ; \qquad B_{4i} = -B_{i4} = B_i , \tag{13.3b}$$

and eliminate the auxiliary fields D, F_i and G_i from the Lagrangian. This now takes the following manifestly O(4) invariant form $(i, j, k, l = 1, \dots 4)$:

$$\begin{aligned}
L = \mathrm{tr}[& -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mathrm{i}\bar{\lambda}_i\gamma^\mu\nabla_\mu\lambda_i + \tfrac{1}{8}\nabla_\mu A_{ij}\nabla^\mu A_{ij} + \tfrac{1}{8}\nabla_\mu B_{ij}\nabla^\mu B_{ij} \\
& + \tfrac{1}{2}\mathrm{i}\bar{\lambda}_i[\lambda_j, A_{ij}] + \tfrac{1}{2}\mathrm{i}\bar{\lambda}_i\gamma_5[\lambda_j, B_{ij}] + \tfrac{1}{32}[A_{ij}, B_{kl}][A_{ij}, B_{kl}] \\
& + \tfrac{1}{64}[A_{ij}, A_{kl}][A_{ij}, A_{kl}] + \tfrac{1}{64}[B_{ij}, B_{kl}][B_{ij}, B_{kl}]] .
\end{aligned} \tag{13.4}$$

The A_{ij} and B_{ij} are self-dual and anti-self-dual tensors of O(4), respectively:

$$A_{ij} = \tfrac{1}{2}\varepsilon_{ijkl}A_{kl}; \qquad B_{ij} = -\tfrac{1}{2}\varepsilon_{ijkl}B_{kl} . \tag{13.5}$$

As in the $N = 2$ case, we argue that the old gluino λ is now part of a vector of O(4) but the gluon is still a singlet. This means that there must be a vector's worth of supersymmetry transformations: $N = 4$. Indeed, the Lagrangian (13.4) is found to transform as a density under the following transformations with four Majorana spinor parameters ζ_i,

$$\begin{aligned}
\delta A_\mu &= \mathrm{i}\bar{\zeta}_i\gamma_\mu\lambda_i \\
\delta A_{ij} &= \bar{\zeta}_i\lambda_j - \bar{\zeta}_j\lambda_i + \varepsilon_{ijkl}\bar{\zeta}_k\lambda_l \\
\delta B_{ij} &= \bar{\zeta}_i\gamma_5\lambda_j - \bar{\zeta}_j\gamma_5\lambda_i - \varepsilon_{ijkl}\bar{\zeta}_k\gamma_5\lambda_l \\
\delta \lambda_i &= -\tfrac{1}{2}\mathrm{i}\sigma^{\mu\nu}\zeta_i F_{\mu\nu} + \mathrm{i}\gamma^\mu\nabla_\mu(A_{ij} + \gamma_5 B_{ij})\zeta_j + \tfrac{1}{2}\mathrm{i}[A_{ij} - \gamma_5 B_{ij}, A_{jk} + \gamma_5 B_{jk}]\zeta_k ,
\end{aligned} \tag{13.6}$$

which close into the on-shell $N = 4$ algebra

$$[\delta^{(1)}, \delta^{(2)}] = 2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\mu\zeta_i^{(2)}\partial_\mu + \delta_{\text{gauge}} + \text{field equations} \tag{13.7a}$$

with the parameter of the gauge transformation given by

$$\alpha = -2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\mu\zeta_i^{(2)}A_\mu + 2\bar{\zeta}_i^{(1)}(A_{ij} + \gamma_5 B_{ij})\zeta_j^{(2)} . \tag{13.7b}$$

Just as for $N = 2$, the internal symmetry group is actually larger than O(N). By defining symplectic Majorana spinors, we could make the somewhat larger symplectic group USp(4) $\approx$ O(5) manifest [61, 62], but by not insisting on any Majorana or pseudo-Majorana condition and going to a chiral notation, we can actually show a yet larger SU(4) symmetry [5]. In order to do this, we break λ_i into chiral parts $\lambda_{\alpha i}$ and $\bar{\lambda}^i_{\dot{\alpha}} \equiv (\lambda_{\alpha i})^\dagger$, and define a complex matrix of scalars

$$M_{ij} \equiv \tfrac{1}{2}(A_{ij} + \mathrm{i}B_{ij}) \tag{13.8}$$

with a reality condition

$$(M_{ij})^\dagger = \tfrac{1}{2}\varepsilon^{ijkl}M_{kl} \equiv M^{ij} . \tag{13.9}$$

The $N = 4$ Lagrangian

$$L = \mathrm{tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda_i\sigma^\mu\nabla_\mu\bar{\lambda}^i + \tfrac{1}{2}\nabla_\mu M_{ij}\,\nabla^\mu M^{ij}\right.$$
$$\left. + i\lambda_i[\lambda_j, M^{ij}] + i\bar{\lambda}^i[\bar{\lambda}^j, M_{ij}] + \tfrac{1}{4}[M_{ij}, M_{kl}][M^{ij}, M^{kl}]\right),$$

(13.10)

is then manifestly SU(4) covariant, as are the transformation laws

$$\delta A_\mu = i\zeta_i\sigma_\mu\bar{\lambda}^i - i\lambda_i\sigma_\mu\bar{\zeta}^i$$
$$\delta M_{ij} = \zeta_i\lambda_j - \zeta_j\lambda_i + \varepsilon_{ijkl}\bar{\zeta}^k\bar{\lambda}^l$$
$$\delta\lambda_i = -\tfrac{1}{2}i\sigma^{\mu\nu}\zeta_iF_{\mu\nu} + 2i\sigma^\mu\,\nabla_\mu M_{ij}\bar{\zeta}^j + 2i[M_{ij}, M^{jk}]\zeta_k.$$

(13.11)

The internal symmetry is SU(4), not U(4). There can be no further U(1), since any phase transformation on M_{ij} is ruled out by the reality condition (13.9), and any phase transformation on λ_i would require one on M_{ij} as well in order to render the Yukawa coupling terms invariant.

In order to make the model interacting at all, the gauge group must be non-Abelian. There is no known off-shell formulation for the non-Abelian theory in terms of unconstrained fields (but compare refs. [61, 62]).

13.2. Finiteness

The fact that bosonic and fermionic loops tend to contribute to quantum infinities with opposite signs has been known since the 1930's. Only in 1974, however, this was effectively put to work for the first time to reduce if not eliminate the problem of quantum divergences: if the fermions and bosons in a model stand in a well defined controlled relationship to each other, i.e., if there is a supersymmetry, certain groups of divergent diagrams will cancel and some of the infinities do not appear. Judging from a footnote in ref. [72], this possibility was pointed out to Wess and Zumino by the late Benjamin Lee, and indeed such cancellations were then found in the Wess–Zumino model.

These cancellations are now understood in terms of a non-renormalisation theorem in superspace: any invariant which can only be written as an integral over chiral superspace but not over the whole of superspace, does not receive any renormalisation corrections [30, 31]. Thus, whereas the kinetic term of the Wess–Zumino model *does* receive such corrections,

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\,\bar{\phi}\phi \to Z_\phi \int d^4x\, d^2\theta\, d^2\bar{\theta}\,\bar{\phi}\phi,$$

(13.12)

the mass and interaction terms *do not*,

$$\frac{m}{2}\int d^4x\, d^2\theta\,\phi^2 + \mathrm{h.c.} \to \frac{m}{2}\int d^4x\, d^2\theta\,\phi^2 + \mathrm{h.c.}$$

(13.13)

$$\frac{g}{3}\int d^4x\, d^2\theta\,\phi^3 + \mathrm{h.c.} \to \frac{g}{3}\int d^4x\, d^2\theta\,\phi^3 + \mathrm{h.c.},$$

(13.14)

so that mass and coupling constant do get renormalised,

$$m \to Z_\phi^{-1} m \quad \text{and} \quad g \to Z_\phi^{-2/3} g ,$$

(13.15)

but, since Z_ϕ is only logarithmically divergent, their renormalisation is only logarithmically divergent. In a generic field theory of spin 0 and $\frac{1}{2}$ particles, without supersymmetry, we would expect a quadratic divergence for the mass and a linear one for the coupling constant (as well as quartic ones for the vacuum energy – these are also absent here since they would break supersymmetry).

Not having gone into the field of superspace perturbation methods (there is an exhaustive book about the subject which contains 1001 formulas and no references [26]) this non-renormalisation theorem for chiral superspace integrals must remain a piece of magic within this report. A second piece of magic will be the non-renormalisation theorem for the factor e^{2gV}. This follows from the background field method for renormalising gauge theories, and has the effect that a super-Yang–Mills model with no matter has only one renormalisation, the logarithmically divergent Z_V, and the coupling constant renormalisation is related to it:

$$g \to Z_V^{-1/2} g .$$

(13.16)

A very early investigation was concerned with the problem of how many matter multiplets could be coupled to a Yang–Mills multiplet before asymptotic freedom was spoiled, i.e., before

$$\beta(g^2) = \mu \frac{\partial}{\partial \mu} Z_g$$

(13.17)

had a positive slope at $g = 0$. Quick calculation showed [18] that up to two chiral multiplets left $\beta'(0)$ negative, that for four and more chiral multiplets it was positive, and that for three chiral multiplets the question could not be decided at the one-loop level: $\beta'(0) = 0$. The condition for asymptotic freedom is that β be monotonically falling at $g = 0$, rather than that $\beta' < 0$; so a higher order calculation was required. It was found that the result depends on more details than just the number of matter multiplets: the representations of the gauge group in which they sit and their ϕ^3 self-coupling term play a role as well.

Specifically, however, it was found that for the $N = 4$ theory (which is, after all, an $N = 1$ theory with three matter multiplets) the β-function remained zero up to three loops [52, 40, 2, 32, 7] and that therefore there were no divergent graphs at all up to that order! This naturally led everyone to suspect that this may actually be a finite field theoretical model in four space–time dimensions, and arguments were sought to proof finiteness to all orders.

These arguments were forthcoming from three different directions. The first was followed by Ferrara and Zumino [20] and by West and myself [64] and exploits the *conformal invariance* of the theory, which will remain a valid symmetry to all orders of perturbation theory if the theory is finite. Conversely, if it is not, there will be a *trace-anomaly* in the energy-momentum tensor – its trace is proportional to the β-function and breaks conformal invariance. Supersymmetry links this trace to the divergence of some axial current, and if there is no *axial anomaly* in $N = 4$ Yang–Mills theory, there cannot be a trace-anomaly either, and the theory is finite. The only axial currents in the model are the ones associated with those nine generators of SU(4) which are not also generators of its real O(4) subgroup. These transform as an irreducible representation **9** under O(4). Neither SU(4) nor O(4) are symmetry groups that are likely to be broken by anomalies. This is particularly clear for the O(4). So even if one of the axial current conservation laws were somehow broken, all nine of them would have to

be, being irreducible under O(4). It was argued that $N = 2$ supersymmetry assured that at least one of the nine would be unbroken (this can probably be shown as well by means of the $USp(4) \approx O(5)$ invariance of the $N = 4$ theory). Therefore there would be no axial current, no trace-anomaly, and no quantum corrections.

The second approach to the problem convinced the theoretical physics community of the finiteness of $N = 4$. Mandelstam [45] and then Brink, Lindgren and Nilsson [6] showed that finiteness was obvious once a particular gauge, the *light-cone gauge*, was chosen for A_μ. The boson-fermion count is manifestly restored in a fixed gauge, and a kind of on-shell superfield formalism can be used to write down possible counterterms. These were found to be absent. A detailed description of this proof cannot be attempted here, since the ground-work for it has not been laid in this report.

The third approach, by Grisaru, Roček and Siegel [32] and by Stelle [69], uses the *non-renormalisation theorem* and the *background field method*. It can be presented here in slightly more detail, as it leads on from eqs. (13.12–16). We know from subsection 12.5 that $N = 2$ supersymmetry demands that the ϕ^3-coupling constant and the gauge-coupling constant must be the same. If $N = 2$ supersymmetry is not to be broken by renormalisation, then this must be true to all orders, and both g's must receive the same corrections. The possible renormalisations of the $N = 1$ gauge theory with superpotential (12.34–36) are

$$
\begin{aligned}
V^2 &\to Z_V V^2; & g_{\text{gauge}} &\to Z_g g_{\text{gauge}} \\
\phi_a^2 &\to Z_\phi (\phi_a)^2; & g_{\phi^3} &\to Z_g' g_{\phi^3} \\
(\phi^a)^2 &\to Z_\phi' (\phi^a)^2; & m &\to m + \delta m \\
(\Psi_a{}^b)^2 &\to Z_\Psi (\Psi_a{}^b)^2
\end{aligned}
\tag{13.18}
$$

with relationships

$$
\begin{aligned}
Z_g^2 Z_V &= 1, & (Z_g')^2 Z_\Psi Z_\phi Z_\phi' &= 1 \\
(1 + \delta m/m)^2 Z_\phi Z_\phi' &= 1
\end{aligned}
\tag{13.19}
$$

between them which follow from the non-renormalisation theorems for e^{2gV} and for chiral superspace integrals.

$N = 2$ supersymmetry requires that the two coupling constants be the same and that the gaugino from V and the spinor from Ψ form a doublet, hence that

$$
Z_g = Z_g'; \qquad Z_V = Z_\Psi \tag{13.20}
$$

which in turn implies, through (13.19), that

$$
Z_\phi Z_\phi' = 1; \qquad \delta m = 0. \tag{13.21}
$$

At this stage ($N = 2$ supersymmetry assumed) there are only three renormalisations left, namely the wave-function renormalisations for the three matter multiplets in the $N = 1$ picture. In the $N = 4$ model, everything is and must be completely symmetric in these three matter multiplets (cf. the obvious O(3) symmetry of the Lagrangian (13.2)). Hence

$$Z_\phi = Z'_\phi = Z_\psi \tag{13.22}$$

and from (3.21) it follows that

$$Z = 1 \quad \text{for all renormalisations} \tag{13.23}$$

and the theory is finite.

This line of argument has been improved in subsequent work by Howe, Stelle and Townsend [36]. A weak point of the original argument was its reliance on $N = 2$ supersymmetry, which is not manifest in (13.2), in order to guarantee eqs. (13.20). It is therefore desirable to write everything in a fully $N = 2$ covariant way. In order to be able to use the superspace arguments, however, it is necessary to formulate it in terms of *unconstrained* $N = 2$ *multiplets*. The hypermultiplet is not suitable since it is subject to the constraint (12.16). By restricting themselves to *real representations* of the gauge group, however, the authors could use the real form of the hypermultiplet. This still has a constraint,

$$[A, \bar{Q}_i] + i[A, \tau_i{}^j \bar{Q}_j] = 0, \tag{13.24}$$

but this could be *relaxed*, replacing the real hypermultiplet with a much more complicated structure, the "relaxed hypermultiplet". This involves not one multiplet with a central charge but rather three of them, without a central charge but with complicated constraint relationships between them which, however, can be solved in terms of *unconstrained prepotential* supermultiplets. The auxiliary field structure is much more complicated than for the constrained hypermultiplet and there are also many "gauge" degrees of freedom, even for the matter sector which involves only spins 0 and $\frac{1}{2}$. The advantages of having unconstrained superfields to work with, and to argue in terms of, are however paramount.

With a real representation of the gauge group and hence with the real form of the hypermultiplets, relaxed or not, the "second SU(2)" is manifest. Since this rotates $\phi_a \to (\phi^a)^\dagger$, the equality of Z_ϕ and Z'_ϕ is guaranteed, and thus, from eq. (13.21),

$$Z_\phi = Z'_\phi = 1 \tag{13.25}$$

even for a theory with only $N = 2$ supersymmetry. The background field formalism when applied to the relaxed hypermultiplet, then showed [37] that indeed the only contributions to the only remaining renormalisation Z_V come from the one-loop graphs. Thus,

If an $N = 2$ theory with matter in a real representation of the gauge group is one-loop finite, it is finite to all orders.

According to the result in ref. [18], this is the case for that particular $N = 2$ theory which has massless matter in the adjoint representation, i.e., for the $N = 4$ model. But it is also true for whole families of $N = 2$ models, some even with mass terms and hence with no conformal invariance [51].

14. Supersymmetry in higher dimensions

The more important models with extended supersymmetry are intimately related to Lagrangian field theories in more than four space–time dimensions. For flat-space supersymmetry this is somewhat of a

mathematical gimmick, but in the case of supergravity, two particular avenues of research have stressed the importance and possible physical significance of these additional dimensions: the "Kaluza–Klein" programme [12] in which the gauge interactions are interpreted as remnants of gravity in additional spatial dimensions, and the "string programme" [28] which only works in ten dimensions in the first place.

It is not possible to write about higher dimensions and their role in supersymmetry without acknowledging the late Joël Scherk. It was he who pioneered the idea of employing higher dimensions to get a grip on the complexities of extended supersymmetry. He co-authored several of the important papers which laid the foundations of the field, in particular Gliozzi, Scherk and Olive [27] and Brink, Scherk and Schwarz [5] where, among other things, the $N = 4$ super-Yang–Mills model was derived from a simple model in ten dimensions, and Cremmer, Julia and Scherk [9] which first employed the eleven-dimensional scenario in the search for the $N = 8$ supergravity which was subsequently worked out in full in a monumental paper by Cremmer and Julia.

As an exercise in getting familiar with higher dimensions, I shall in this section cover some of the grounds of ref. [5], and show how $N = 2$ and $N = 4$ supersymmetric Yang–Mills theories fit beautifully into a world of six and ten dimensions, respectively.

14.1. Spinors in higher dimensions

Before we can talk about supersymmetry in higher dimensions, we must know what spinors are in higher dimensions. Supersymmetry, after all, is by definition a symmetry which involves spinorial as well as tensorial quantities. The finite dimensional representations of the Lie algebras of pseudo-orthogonal groups $O(d_+, d_-)$ fall into two classes: some representations are contained in multiple direct products of the fundamental vector representation of the group and others are not. The former are the *tensor representations*, the latter are the *spinor representations*.* An easy handle on the spinor representations is provided by the *Dirac matrices* and their properties. These are irreducible representations of the Dirac algebra

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \cdot 1 \ , \tag{14.1}$$

from which it follows that the matrices $\frac{1}{2}\Sigma_{ab}$ with

$$\Sigma_{ab} \equiv i\Gamma_{ab} \equiv i\Gamma_{[a}\Gamma_{b]} \tag{14.2}$$

represent the algebra of the Lorentz group in d dimensions with metric η_{ab};

$$\frac{1}{2}\Sigma_{ab} = \Sigma(M_{ab})$$

$$[M_{ab}, M_{cd}] = i(\eta_{bc}M_{ad} - \eta_{bd}M_{ac} - \eta_{ac}M_{bd} + \eta_{ad}M_{bc}) \ . \tag{14.3}$$

In appendix A.7, which contains many important definitions and must be seen as an integral part of the present section, we learn that the (complex) matrix dimension of this representation is

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* With respect to their global properties, these are very different: the tensor representations of the algebra indeed generate representations of the group $O(d_+, d_-)$, the spinor representations generate those of the covering group spin(d).

$$n = 2^{d/2} \qquad \text{for even dimension } d$$

$$n = 2^{(d-1)/2} \qquad \text{for odd dimension } d,$$

(14.4)

which grows exponentially with d. Spinor representations are thus most certainly not "small". Their increasing size actually puts limits on the largest dimensions which can possibly sustain supersymmetry at all: since the total number of chiral spinor charges in four dimensions is limited to $N = 4$ for renormalisable models and to $N = 8$ for supergravity, the total number of real spinorial charges must not exceed 16 or 32, respectively.

It is therefore important to establish the *smallest possible dimension* of a spinor representation. Equation (14.4) gives the dimension as $2n$ (a factor of two because n is the number of complex dimensions). Two ways are available to reduce that number, *chirality* conditions and *reality* conditions.

Chirality conditions are possible for *even dimensions*, where the matrix Γ_{d+1} is non-trivial. It commutes with the Σ_{ab} which implies that the two projections

$$\tfrac{1}{2}\Sigma_{ab}^{\pm} = \tfrac{1}{2}(1 \pm \sqrt{\beta}\,\Gamma_{d+1})\tfrac{1}{2}\Sigma_{ab}$$

(14.5)

(see table A.2 in the Appendix for the values of β) are also representations of the algebra (14.3). Since these *chiral representations* involve the projection operators $\tfrac{1}{2}(1 \pm \sqrt{\beta}\,\Gamma_{d+1})$, the number of dimensions of the representation space has been halved. This is not possible for odd dimensions where $\Gamma_{d+1} \propto 1$.

A reality condition (*Majorana condition*) for a spinor will have the general form

$$\psi = X\psi^*$$

(14.6a)

with X some non-singular $n \times n$ matrix and $*$ the component-wise conjugate. Since an infinitesimal Lorentz transformation acts on a spinor as

$$\delta\psi = \tfrac{1}{4}i\lambda^{ab}\Sigma_{ab}\psi = -\tfrac{1}{4}\lambda^{ab}\Gamma_{ab}\psi$$

and on the complex conjugate as

$$\delta\psi^* = -\tfrac{1}{4}\lambda^{ab}\Gamma_{ab}^*\psi^*,$$

the matrix X must have the property

$$\Gamma_{ab}^* = X^{-1}\Gamma_{ab}X$$

in order to make the Majorana condition (14.6a) Lorentz covariant. The only matrices which do this are multiples of D and $\tilde{D}$ as defined in eqs. (A.60) and (A.64):

$$X = D \qquad \text{or} \qquad X = \Gamma_{d+1}^{-1}D \equiv \tilde{D}.$$

(14.6b)

Not in all cases, however, will this be consistent. Only if at least one of the conditions

$$DD^* = \delta = +1 \qquad \text{or} \qquad \tilde{D}\tilde{D}^* = \tilde{\delta} = +1$$

(14.7)

is fulfilled, can we have a Majorana condition (take the conjugate of eq. (14.6) and insert). Thus table A.4 of the Appendix, which gives the signs of δ and $\tilde{\delta}$, acquires additional importance since it shows for which space–time dimensions and metrics Majorana conditions are possible: at least one plus sign in the entry in the table is required.

A standard way of writing Majorana conditions is

$$\psi = \psi^c \equiv C\bar{\psi}^T \tag{14.8}$$

(or $\psi = \Gamma_{d+1}^{-1}\psi^c$) with

$$\bar{\psi} \equiv \psi^\dagger A. \tag{14.9}$$

Finally, the question arises under which circumstances a chiral spinor can also satisfy a Majorana condition. For this, we must have

$$(\mathbb{1} \pm \sqrt{\beta}\, \Gamma_{d+1})\psi = D(\mathbb{1} \pm \sqrt{\beta}^*\, \Gamma_{d+1}^*)\psi^*$$

or the corresponding equation with $\tilde{D}$. We evaluate this and find

$$= D\psi^* \pm \sqrt{\beta}^* D\Gamma_{d+1}^* D^{-1}D\psi^* = (\mathbb{1} \pm \sqrt{\beta}^*\Gamma_{d+1})D\psi^*.$$

Therefore we must have $\sqrt{\beta}$ real, i.e. $\beta = +1$, and $\delta = +1$. This implies that

$$\delta = \tilde{\delta} = +1 \tag{14.10}$$

because of eq. (A.66). Starting with $\tilde{D}$ would have led to the same result. We see that *chiral Majorana spinors* exist exactly for those even dimensional space–times where the entry in table A.4 is $++$.

I summarise the results in table 14.1. Clearly, the maximal dimension in which $N = 4$ Yang–Mills theory can be expressed is $d = 10$, and the maximal dimension for $N = 8$ supergravity is $d = 11$. $N = 2$ supersymmetry, with eight spinorial charges, seems to want to live in six dimensions. We consider this case first.

Table 14.1
Chirality and reality of spinors in Minkowski space–times with $d \leq 12$

d	1	2	3	4	5	6	7	8	9	10	11	12
n	1	2	2	4	4	8	8	16	16	32	32	64
Chiral spinors	—	yes	—	yes	—	yes	—	yes	—	yes	—	yes
Real spinors	yes	yes	yes	yes	—	—	—	yes	yes	yes	yes	yes
Chiral real spinors	—	yes	—	—	—	—	—	—	—	yes	—	—
Minimal spinor dimension	1	1	2	4	8	8	16	16	16	16	32	64

14.2. Supersymmetric Yang–Mills theory in d = 6

Consider the following Lagrangian in a six dimensional Minkowski space for a gauge field A_a ($a = 0, \ldots, 3, 5, 6$) and a chiral spinor λ in the adjoint representation of the gauge group:

$$L = \mathrm{tr}(-\tfrac{1}{4}F_{ab}F^{ab} + \tfrac{1}{2}\mathrm{i}\bar{\lambda}\Gamma^a \overset{\leftrightarrow}{\nabla}_a\lambda) \tag{14.11}$$

$$\lambda = \tfrac{1}{2}(1 - \Gamma_7)\lambda \tag{14.12}$$

$$\nabla_a\lambda = \partial_a\lambda + \mathrm{i}[A_a, \lambda]. \tag{14.13}$$

A particular representation of the Dirac matrices is:

$$\Gamma_\mu = \begin{bmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{bmatrix} \quad \text{for } \mu = 0, \ldots 3$$

$$\Gamma_5 = \begin{bmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{bmatrix}; \quad \Gamma_6 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{14.14}$$

$$\Gamma_7 = \Gamma_0 \cdots \Gamma_6 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad A = \Gamma_0.$$

First we show the relationship between the Lagrangian (14.11) and $d = 4$, $N = Z$ supersymmetry. The chiral spinor λ can be written as

$$\lambda = \begin{bmatrix} \chi \\ 0 \end{bmatrix} \tag{14.15}$$

with χ an unconstrained, complex 4-spinor. We can then rewrite

$$\frac{\mathrm{i}}{2}\bar{\lambda}\Gamma^a \overset{\leftrightarrow}{\nabla}_a\lambda = \frac{\mathrm{i}}{2}\bar{\chi}\gamma^\mu \overset{\leftrightarrow}{\nabla}_\mu\chi - \frac{\mathrm{i}}{2}\bar{\chi}\gamma_5 \overset{\leftrightarrow}{\nabla}_5\chi - \frac{\mathrm{i}}{2}\bar{\chi}\overset{\leftrightarrow}{\nabla}_6\chi.$$

As a next step, we assume that nothing depends on x^5 and x^6,

$$\partial_5 = \partial_6 = 0. \tag{14.16}$$

This is called *trivial dimensional reduction*. It gives us a gauge covariant transformation law for A_5 and A_6 (because $\partial_5\alpha = \partial_6\alpha = 0$) and

$$\nabla_{5,6}\chi = \mathrm{i}[A_{5,6}, \chi]$$

$$F_{\mu5} = \nabla_\mu A_5, \qquad F_{\mu6} = \nabla_\mu A_6 \tag{14.17}$$

$$F_{56} = \mathrm{i}[A_5, A_6].$$

The Lagrangian (14.11) now depends only on four coordinates x^μ and reads

$$L = \mathrm{tr}(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\nabla_\mu A_5 \nabla^\mu A_5 + \tfrac{1}{2}\nabla_\mu A_6 \nabla^\mu A_6 + \tfrac{1}{2}i\,\bar{\chi}\gamma^\mu \overleftrightarrow{\nabla}_\mu \chi$$

$$- \bar{\chi}[\chi, A_6] - \bar{\chi}\gamma_5[\chi, A_5] + \tfrac{1}{2}[A_5, A_6]^2). \tag{14.18}$$

With the identifications

$$\chi = \frac{1}{\sqrt{2}}(\lambda_1 - i\lambda_2)$$

$$A_5 = N, \qquad A_6 = M, \tag{14.19}$$

this is exactly the $N = 2$ Lagrangian in $O(2)$ notation, eq. (12.4). In the calculation, a number of terms vanish due to symmetry properties of bi-spinors.

The $N = 2$, $d = 4$ Lagrangian can thus be derived from a $d = 6$ Lagrangian by trivial dimensional reduction. Since the former is a density under the set of supersymmetry transformations (12.6), we can use these as a starting point and try to construct supersymmetry transformations in six dimensions under which (14.11) is a density. We first define a spinor parameter ζ from the two Majorana spinor parameters ζ_i of the four-dimensional transformations. In analogy with (14.15) and (14.19) this is

$$\zeta = \frac{1}{\sqrt{2}}\begin{bmatrix} \zeta_1 - i\zeta_2 \\ 0 \end{bmatrix}, \tag{14.20}$$

and we evaluate

$$i\bar{\zeta}\Gamma_\mu\lambda = \tfrac{1}{2}i\bar{\zeta}_i\gamma_\mu\lambda_i + \tfrac{1}{2}\varepsilon_{ij}\bar{\zeta}_i\gamma_\mu\lambda_j$$

$$i\bar{\zeta}\Gamma_5\lambda = \tfrac{1}{2}i\bar{\zeta}_i\gamma_5\lambda_i + \tfrac{1}{2}\varepsilon_{ij}\bar{\zeta}_i\gamma_5\lambda_j \tag{14.21}$$

$$i\bar{\zeta}\Gamma_6\lambda = \tfrac{1}{2}i\bar{\zeta}_i\lambda_i + \tfrac{1}{2}\varepsilon_{ij}\bar{\zeta}_i\lambda_j.$$

Using the reality properties for the terms on the right-hand side, we can combine the δA_μ, δM and δN of eqs. (12.6) into a single equation

$$\delta A_a = i\bar{\zeta}\Gamma_a\lambda - i\bar{\lambda}\Gamma_a\zeta.$$

The transformation law for λ can be calculated from $\delta\lambda_i$ and is

$$\delta\lambda = -\tfrac{1}{2}i\Sigma^{\mu\nu}\zeta F_{\mu\nu} - i\Sigma^{\mu 5}\zeta \nabla_\mu A_5 - i\Sigma^{\mu 6}\zeta \nabla_\mu A_6 + \Sigma^{56}\zeta[A_5, A_6].$$

The dimensionally reduced Lagrangian (14.18) is a density under these transformations. We expect, and indeed verify, that the full Lagrangian (14.11) is a density under the obvious generalisation of this to $\partial_5 \neq 0 \neq \partial_6$:

$$\delta A_a = i\bar{\zeta}\Gamma_a\lambda - i\bar{\lambda}\Gamma_a\zeta$$

$$\delta\lambda = -\tfrac{1}{2}i\Sigma^{ab}\zeta F_{ab}. \tag{14.22}$$

A slightly more involved calculation will give the commutator of two supersymmetry transformations in a form reminiscent of eq. (12.14):

$$[\delta^{(1)}, \delta^{(2)}] = 2i(\bar{\zeta}^{(1)}\Gamma^a\zeta^{(2)} - \bar{\zeta}^{(2)}\Gamma^a\zeta^{(1)})\partial_a + \delta_{\text{gauge}} + \text{eq. of motion} \tag{14.23a}$$

with a field-dependent parameter for the gauge transformation,

$$\alpha = -2i(\bar{\zeta}^{(1)}\Gamma^a\zeta^{(2)} - \bar{\zeta}^{(2)}\Gamma^a\zeta^{(1)})A_a. \tag{14.23b}$$

All of this is consistent with the fact that the $N = 2$ supersymmetry algebra with two central charges in four dimensions,

$$\{Q_i, \bar{Q}_j\} = 2\delta_{ij}\gamma^\mu P_\mu + 2i\varepsilon_{ij}Z + 2i\varepsilon_{ij}\gamma_5 Z'$$

$$[Q_i, P_\mu] = [Q_i, Z] = [Q_i, Z'] = 0 \tag{14.24}$$

$$[P_\mu, P_\nu] = [P_\mu, Z] = [P_\mu, Z'] = [Z, Z'] = 0,$$

can be recast into a six-dimensional form

$$\{Q, \bar{Q}\} = (\mathbb{1} + \Gamma_7)\Gamma^a P_a, \qquad \{Q, Q\} = 0$$

$$[Q, P_a] = 0, \qquad [P_a, P_b] = 0 \tag{14.25}$$

with

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ Q_1 - iQ_2 \end{pmatrix}; \qquad P_5 = -Z'; \qquad P_6 = -Z. \tag{14.26}$$

In this subsection, we have seen that there is a very close relationship indeed between $N = 2$ super-Yang–Mills theory in $d = 4$ and simple super-Yang–Mills theory in $d = 6$. The transition from the latter to the former is achieved by the trivial dimensional reduction $\partial_5 = \partial_6 = 0$. The next subsection will be devoted to the matter multiplet in $d = 6$ and its behaviour under dimensional reduction.

14.3. The hypermultiplet in $d = 6$

The following Lagrangian for two complex scalars A and B and an *anti*-chiral spinor $\psi = \frac{1}{2}(\mathbb{1} + \Gamma_7)\psi$ in six dimensions,

$$L = \partial_a A^\dagger \partial^a A + \partial_a B^\dagger \partial^a B + \frac{1}{4}i\bar{\psi}\Gamma^a \overleftrightarrow{\partial}_a\psi \tag{14.27}$$

is a density under a set of supersymmetry transformations

$$\delta A = \bar{\zeta}\psi, \qquad \mathrm{d}B = \overline{\zeta^c}\psi$$

$$\delta\psi = -i\Gamma^a \partial_a(A\zeta + B\zeta^c) \tag{14.28}$$

which have a closed algebra of the form

$$[\delta^{(1)}, \delta^{(2)}] = 2\mathrm{i}(\bar{\zeta}^{(1)}\Gamma^a\zeta^{(2)} - \bar{\zeta}^{(2)}\Gamma^a\zeta^{(1)})\partial_a + \text{eq. of motion} . \tag{14.29}$$

We could "gauge-covariantise" this in order to make the hypermultiplet interact with the gauge multiplet of the previous subsection. But since I haven't given the off-shell formalism for the gauge multiplet, this would involve changes not only in the Lagrangian (14.27) and the transformation laws (14.28) but also in (14.22). As this is unnecessarily complicated and not very instructive, it may just be said that it can be done, without any effect on the more important aspect that I now wish to discuss, namely *non-trivial* dimensional reduction on a *torus*.

Clearly, a trivial dimensional reduction $\partial_5 = \partial_6 = 0$ would reduce the Lagrangian (14.27) to the $d = 4$ Lagrangian for a free, massless hypermultiplet. But there is another way of obtaining a sensible theory in four dimensions from (14.27): we assume that all fields are *periodic* in x^5 and x^6 with periods $1/m'$ and $1/m$, respectively. At each point in four-dimensional space–time, the remaining two dimensions then have the shape of a donut,
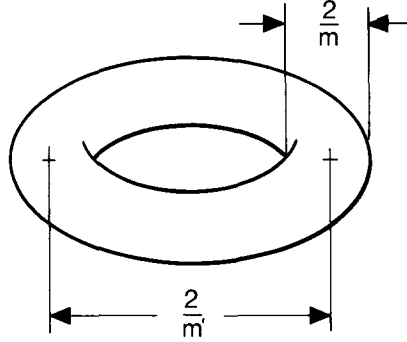
and we can Fourier decompose the fields, e.g.,

$$A(x^\mu, x^5, x^6) = \sum_{nn'} \exp(-\mathrm{i}n'm'x^5 - \mathrm{i}nmx^6)\, A_{nn'}(x^\mu) . \tag{14.30}$$

The Lagrangian will become the sum of Lagrangians for the Fourier components,

$$L_{nn'} = \partial_\mu A^\dagger_{nn'}\, \partial^\mu A_{nn'} + \partial_\mu B^\dagger_{nn'}\, \partial^\mu B_{nn'} + \tfrac{1}{4}\mathrm{i}\bar{\psi}_{nn'}\overleftrightarrow{\partial}\psi_{nn'}$$

$$- (n'^2 m'^2 + n^2 m^2)\, (A^\dagger_{nn'} A_{nn'} + B^\dagger_{nn'} B_{nn'}) - \frac{n'm'}{2}\, \bar{\psi}_{nn'}\gamma_5\psi_{nn'} + \frac{nm}{2}\, \bar{\psi}_{nn'}\psi_{nn'} . \tag{14.31}$$

The fields $\psi_{nn'}$ are the Dirac spinors in four dimensions which form the bottom halves of the anti-chiral spinors $\psi_{nn'}$ in six dimensions.

We see that our assumption that the "world" is a torus in the fifth and sixth dimension led to a spectrum with an infinite tower of ever more massive hypermultiplets. Their masses are, after a γ_5-transformation on ψ:

$$M_{nn'} = \sqrt{n^2 m^2 + n'^2 m'^2} \qquad \text{with } n, n' \text{ integers} . \tag{14.32}$$

Apart from the massless mode, the lowest mass is $M = m'$ at $n = 0$, $n' = 1$ if we assume that $m' \leq m$.

The standard massive hypermultiplet of section 12 is the case where only one (massive) mode is present in the Fourier decomposition. Having identified the central charges as $-P_5$ and $-P_6$ in the previous subsection, we see that the "multiplet shortening condition" (12.15), which in general reads

$$\sum_{\substack{\text{central} \\ \text{charges}}} Z_i^2 = P_\mu P^\mu , \tag{14.33}$$

is just the higher dimensional *massless equation of motion*

$$\partial_a \partial^a = 0 , \tag{14.34}$$

and the equation of motion in four dimensions,

$$\delta_z \phi = \mathrm{i} m \phi , \tag{14.35}$$

is just the *periodicity condition*

$$\frac{\partial}{\partial x^6} \phi = -\mathrm{i} m \phi , \tag{14.36}$$

i.e., the condition that the world be a torus in the sixth dimension (section 12 assumed $m' = 0$ so that the torus had infinite radius in the fifth dimension).

14.4. Supersymmetric Yang–Mills theory in $d = 10$

A quick glance at table 14.1 will show that the maximal dimension which can sustain $N = 4$ supersymmetry (16 real spinorial charges) is $d = 10$. A 16-component spinor in *ten-dimensional Minkowski space* must be both chiral,

$$\lambda = \tfrac{1}{2}(\mathbb{1} - \Gamma_{11})\lambda , \tag{14.37}$$

and Majorana,

$$\lambda = C\bar{\lambda}^{\mathrm{T}} . \tag{14.38}$$

Such a spinor field will describe eight fermionic degrees of freedom, and we can again hope that a λ which is minimally coupled to a gauge field (which has $d - 2$ degrees of freedom, also 8) may be a supersymmetric system, just as it happened in the case of $d = 6$.

Indeed, the Lagrangian

$$L = \mathrm{tr}(-\tfrac{1}{4}F_{ab}F^{ab} + \tfrac{1}{2}\mathrm{i}\bar{\lambda}\Gamma^a \nabla_a \lambda) \tag{14.39}$$

is a density under supersymmetry transformations

$$\delta A_a = i\bar{\zeta}\Gamma_a\lambda, \qquad \delta\lambda = -\tfrac{1}{2}i\Sigma^{ab}\zeta F_{ab} \tag{14.40}$$

whose parameter ζ is also a chiral Majorana spinor. The algebra of these transformations is

$$[\delta^{(1)}, \delta^{(2)}] = 2i\bar{\zeta}^{(1)}\Gamma^a\zeta^{(2)}\partial_a + \delta_{\text{gauge}} + \text{eq. of motion} \tag{14.41a}$$

with a field-dependent parameter for the gauge transformation,

$$\alpha = -2i\bar{\zeta}^{(1)}\Gamma^a\zeta^{(2)}A_a . \tag{14.41b}$$

In working out the algebra, it is necessary to perform a Fierz transformation, take account of the chirality properties of $\zeta^{(i)}$, of the symmetry properties of $\Gamma_a C$, $\Gamma_{abc}C$ and $\Gamma_{abcde}C$, and finally of the second of eqs. (A.70) for $k = 1$ and $k = 5$.

To obtain a theory in four dimensions from this, we have to perform a trivial dimensional reduction:

$$\partial_{4+m} = 0 \qquad \text{for } m = 1, \ldots 6 . \tag{14.42}$$

Reduction on a torus would lead to massive spin-1 modes and thus to a non-renormalisable theory. Indeed, while this particular model is the renormalisable, conformally invariant and finite $N = 4$ super-Yang–Mills theory in four dimensions, it doesn't have any of these nice properties in ten.

We expect the six components $A_5, \ldots A_{10}$ to become scalars and the 16 components of λ to break up into four chiral spinors $\lambda_{\alpha i}$. The Lorentz symmetry group $O(1,9)$ of the $d = 10$ theory will break down to $O(1,3) \otimes O(6) = SL(2, c) \otimes SU(4)$. Therefore we can expect to end up with the $SU(4)$-covariant version of the $N = 4$ super-Yang–Mills model.

The details of the dimensional reduction are a rather involved exercise in higher-dimensional Dirac matrix algebra. Because of their similarity to the dimensional reductions of supergravity models, I shall nevertheless present them here.

We must first consider a particular representation of the $d = 10$ Dirac matrices. We choose

$$\Gamma_\mu = \gamma_\mu \otimes 1 \qquad \text{for } \mu = 0, \ldots 3$$

$$\Gamma_{4+m} = \gamma_5 \otimes \tilde{\Gamma}_m \quad \text{for } m = 1, \ldots 6 \tag{14.43}$$

with γ_μ, γ_5 the standard 4×4 Dirac matrices in four-dimensional Minkowski space and $\tilde{\Gamma}_m$ a set of 8×8 Dirac matrices for *six-dimensional Euclidean space* with $\eta_{mn} = +\delta_{mn}$. We also have matrices A, $C_{(10)}$ and Γ_{11}:

$$A = \Gamma_0$$

$$C_{(10)} = C_{(4)} \otimes \tilde{C}_{(6)} \tag{14.44}$$

$$\Gamma_{11} = \gamma_0 \cdots \gamma_3(\gamma_5)^6 \otimes \tilde{\Gamma}_1 \cdots \tilde{\Gamma}_6 = -\gamma_5 \otimes \tilde{\Gamma}_7 .$$

A representation for the $\tilde{\Gamma}_m$ could be obtained from the Γ_8 for six-dimensional Minkowski space, see eqs.

(14.14), by the substitutions

$$\tilde{\Gamma}'_4 = \Gamma_0; \qquad \tilde{\Gamma}'_m = \mathrm{i}\Gamma_m \qquad \text{for } m = 1, 2, 3, 5, 6. \tag{14.45}$$

It will, however, be much more convenient to work with a set of $\tilde{\Gamma}_m$ which are obtained from these by a similarity transformation

$$\tilde{\Gamma}_m = S^{-1}\tilde{\Gamma}'_m S \qquad \text{with} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_5 C \end{bmatrix}. \tag{14.46}$$

The result for the new $\tilde{\Gamma}_m$ is

$$\tilde{\Gamma}_m = \begin{bmatrix} 0 & \tilde{\sigma}_m \\ \tilde{\sigma}_m^{-1} & 0 \end{bmatrix} \tag{14.47a}$$

with

$$\tilde{\sigma}_1 = \mathrm{i}\gamma_1\gamma_5 C; \qquad \tilde{\sigma}_2 = \mathrm{i}\gamma_2\gamma_5 C; \qquad \tilde{\sigma}_3 = \mathrm{i}\gamma_3\gamma_5 C$$
$$\tilde{\sigma}_4 = \gamma_0\gamma_5 C; \qquad \tilde{\sigma}_5 = -\mathrm{i}C; \qquad \tilde{\sigma}_6 = -\mathrm{i}\gamma_5 C. \tag{14.47b}$$

Here C and the γ's are the standard $O(1, 3)$ matrices whose explicit form is given in appendix A.5. All six matrices $\tilde{\sigma}_m$ are antisymmetric. Another property of the $\tilde{\sigma}_m$ will later be of interest: the explicit form of $\tilde{\sigma}_1$

$$\tilde{\sigma}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

satisfies the equations $(\tilde{\sigma}_{1ij})^* = -(\tilde{\sigma}_1^{-1})^{ij} = \frac{1}{2}\varepsilon^{ijkl}\tilde{\sigma}_{1kl}$. SU(4) covariance (or explicit calculation) ensures this for all $\tilde{\sigma}_m$ and we have

$$(\tilde{\sigma}_m^*)^{ij} = -(\tilde{\sigma}_m^{-1})^{ij} = \frac{1}{2}\varepsilon^{ijkl}(\tilde{\sigma}_m)_{kl}. \tag{14.48}$$

The explicit forms of the matrices $\tilde{C}_{(6)}$ and $\tilde{\Gamma}_7$ are straightforward to evaluate:

$$\tilde{C}_{(6)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{14.49}$$

$$\tilde{\Gamma}_7 = \mathrm{i}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{14.50}$$

A general 32-component complex spinor would look like

$$\lambda = \begin{pmatrix} \lambda_{\alpha i} \\ \chi_\alpha^i \\ \bar{\omega}_i^{\dot\alpha} \\ \bar{\psi}^{\dot\alpha i} \end{pmatrix} \quad \text{with } \alpha, \dot\alpha = 1, 2 \text{ and } i = 1, \ldots 4 . \tag{14.51}$$

In this notation, the matrices A, $C_{(10)}$ and Γ_{11} are now determined as

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ; \quad \Gamma_{11} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{14.52a}$$

$$C_{(10)} = \begin{pmatrix} 0 & -\varepsilon_{\alpha\beta}\delta_i^{\ j} & 0 & 0 \\ -\varepsilon_{\alpha\beta}\delta_j^{\ i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon^{\dot\alpha\dot\beta}\delta_i^{\ j} \\ 0 & 0 & -\varepsilon^{\dot\alpha\dot\beta}\delta_j^{\ i} & 0 \end{pmatrix} \tag{14.52b}$$

which implies

$$\bar{\lambda} = (\omega^{\alpha i}, \psi_i^\alpha, \bar{\lambda}_{\dot\alpha}^i, \bar{\chi}_{\dot\alpha i}) \quad \text{and} \quad \lambda^c = \begin{pmatrix} \psi_{\alpha i} \\ \omega_\alpha^i \\ \bar{\chi}_i^{\dot\alpha} \\ \bar{\lambda}^{\dot\alpha i} \end{pmatrix} . \tag{14.53}$$

The chirality condition (14.37) is now

$$\chi_\alpha^i = \bar{\omega}_i^{\dot\alpha} = 0 , \tag{14.54}$$

and the Majorana condition (14.38) is

$$\bar{\psi}^{\dot\alpha i} = \bar{\lambda}^{\dot\alpha i} \equiv (\lambda_{\beta i})^\dagger \varepsilon^{\dot\beta\dot\alpha} ; \qquad \bar{\omega}_i^{\dot\alpha} = \bar{\chi}_i^{\dot\alpha} \equiv (\chi_\beta^i)^\dagger \varepsilon^{\dot\beta\dot\alpha} . \tag{14.55}$$

We see that the surviving 16 components of λ are just four chiral two-spinors $\lambda_{\alpha i}$ in the representation **4** of SU(4).

We proceed to decompose the terms in the $d = 10$ Lagrangian (14.39) under the assumption of trivial dimensional reduction, eq. (14.42):

$$-\tfrac{1}{4}F_{ab}F^{ab} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\nabla_\mu M_m \nabla^\mu M_m + \tfrac{1}{4}[M_m, M_n]^2 \tag{14.56a}$$

$$\tfrac{1}{2}i\bar{\lambda}\Gamma^a \nabla_a\lambda = \tfrac{1}{2}i\bar{\lambda}\Gamma^\mu \nabla_\mu\lambda - \tfrac{1}{2}\bar{\lambda}\Gamma_{4+m}[\lambda, M_m]$$

$$= \tfrac{1}{2}i\lambda_i\sigma^\mu \overleftrightarrow{\nabla}_\mu\bar{\lambda}^i + \tfrac{1}{2}i\lambda_i[\lambda_j, (\tilde{\sigma}_m^{-1})^{ij}M_m] - \tfrac{1}{2}i\bar{\lambda}^i[\bar{\lambda}^j, (\tilde{\sigma}_m)_{ij}M_m] . \tag{14.56b}$$

In the last step we have made the transition from $d = 10$ covariant notation to $d = 4$ chiral two-spinors.

The scalar fields M_m are the last six components of A_a:

$$M_m \equiv A_{4+m} \qquad \text{for } m = 1, \ldots 6, \tag{14.57}$$

and if we define

$$M_{ij} \equiv -\tfrac{1}{2}(\tilde{\sigma}_m)_{ij} M_m, \qquad M^{ij} \equiv \tfrac{1}{2}(\tilde{\sigma}_m^{-1})^{ij} M_m \tag{14.58}$$

we get the relationship (13.9) because of (14.48) and the reality of M_m. Since there is also a trace-orthogonality relationship for the $\tilde{\sigma}_m$,

$$\text{tr } \tilde{\sigma}_m \tilde{\sigma}_n^{-1} = 4 \delta_{mn}, \tag{14.59}$$

we can write

$$M_m M_m = M_{ij} M^{ij}. \tag{14.60}$$

Using this on (14.56) gives us, up to a divergence, exactly the SU(4)-covariant $N = 4$ Lagrangian (13.10). Clearly, the $d = 4$, $N = 4$ transformation laws (13.11) can be obtained from (14.40) in a similar way.

15. The supercurrent

At the end of section 5, I introduced a supercurrent for the Wess–Zumino model. This was a local Majorana spinor-vector current $J_{\mu\alpha}$, constructed from the fields of the model and conserved if those fields were subject to their equations of motion. In the present section, I shall present a more detailed study of the supercurrent, of its properties and of the multiplet to which it belongs [19].

15.1. The multiplet of currents

The time component of the supercurrent, when integrated over d^3x, gives rise to the supersymmetry charges Q_α, as already stated in eq. (5.10):

$$Q_\alpha = \int d^3x \, J_{0\alpha}(x). \tag{15.1}$$

The time-independence of Q is, as usual, guaranteed by the conservation law for J_μ and the assumed vanishing of surface integrals. As an x-dependent field, the supercurrent cannot stand alone but must be part of a supermultiplet. The task at hand is to determine that multiplet.

From the general definition of the supersymmetry transformations,

$$\delta J_\mu = -i[J_\mu, \bar{\zeta}Q],$$

we get by integration over d^3x

$$\int d^3x \, \delta J_0 = -\mathrm{i}[Q, \bar{Q}\zeta] = -2\mathrm{i}\gamma^\mu \zeta P_\mu \,.$$

The momentum four-vector, on the other hand, is the d^3x integral over components of the *energy-momentum tensor* (*stress tensor*):

$$P_\mu = \int d^2x \, \theta_{0\mu} \,, \tag{15.2}$$

and we see that the algebra of supersymmetry itself already imposes a very intimate relationship between the supercurrent and the stress tensor. This relationship originates from the intimate relationship between the energy-momentum vector and the supersymmetry charge itself. It means that both currents must be in the same supermultiplet and is of great importance for supergravity: Einstein's equation states that the density of energy and momentum is the source of the gravitational field. As this density is related to the super-charge density $J_{\mu\alpha}$, we expect the gravitational field to be related to some other field whose source is $J_{\mu\alpha}$. This will be the *gravitino field* $\psi_{\mu\alpha}$, the field of the superpartner of the graviton. The gravitino has zero mass for unbroken supersymmetry and always spin $\frac{3}{2}$.

In order to find the full supercurrent multiplet, we can start from the old expression (5.9) for J_μ,

$$J_\mu^{\mathrm{old}} = \partial\!\!\!/(A - \gamma_5 B)\gamma_\mu \psi + \mathrm{i}m\gamma_\mu(A - \gamma_5 B)\psi + \mathrm{i}g\gamma_\mu(A - \gamma_5 B)^2 \psi \,,$$

and calculate its supervariation, using the transformation rules (5.6) for the fields A, B and ψ as well as their equations of motion (5.5). The matter fields are thus taken to be on-shell. The result of the calculation will be an utter mess. This is due to the fact that the chosen form of J_μ does not have nice conformal properties. This point will become clearer later, for the time being it may suffice to say that the particular "improvement term" of subsection 5.5 made J_μ^{old} look rather pretty, but it does not lead to a reasonably simple current multiplet. A "nice" supercurrent will have the property that, at the classical level,

$$\gamma^\mu J_\mu = 0 \qquad \text{if} \qquad m = 0 \,,$$

i.e. that the "γ-trace" of J_μ vanishes for zero mass. The γ-trace is the component $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ in the decomposition of a vector-spinor $(\frac{1}{2}, \frac{1}{2}) \otimes ((\frac{1}{2}, 0) \oplus (0, \frac{1}{2}))$. For the old supercurrent we get instead

$$\gamma^\mu J_\mu^{\mathrm{old}} = -2\partial\!\!\!/[(A + \gamma_5 B)\psi] + 2\mathrm{i}m(A - \gamma_5 B)\psi \,,$$

with the first term violating the nice-ness postulate. We can, however, *improve J_μ* by

$$J_\mu = J_\mu^{\mathrm{old}} - \frac{2\mathrm{i}}{3} \partial^\nu[\sigma_{\mu\nu}(A + \gamma_5 B)\psi] \,.$$

As stated in subsection 5.5, such improvements by quantities which are preserved independent of field equations are always possible.

We proceed and calculate the supersymmetry variations first of the improved J_μ and then of the fields which appear in δJ_μ and so on until the multiplet closes. The result of this rather non-trivial

calculation is the following *multiplet of currents*:

$$\delta X = -\tfrac{1}{3}\mathrm{i}\bar{\zeta}\gamma^\mu J_\mu, \qquad \delta Y = -\tfrac{1}{3}\mathrm{i}\bar{\zeta}\gamma_5\gamma^\mu J_\mu$$

$$\delta j_\mu^{(5)} = -\bar{\zeta}\gamma_5 J_\mu + \tfrac{1}{3}\bar{\zeta}\gamma_5\gamma_\mu\gamma^\nu J_\nu$$

$$\delta J_\mu = -2\mathrm{i}\gamma^\nu\zeta\theta_{\mu\nu} + \mathrm{i}\gamma^\nu\gamma_5\zeta(\partial_\nu j_\mu^{(5)} - \eta_{\mu\nu}\partial^\rho j_\rho^{(5)}) + \tfrac{1}{2}\mathrm{i}\varepsilon_\mu{}^{\nu\rho\sigma}\gamma_\nu\zeta\partial_\rho j_\sigma^{(5)} - \mathrm{i}\sigma_{\mu\nu}\partial^\nu(X + \gamma_5 Y)\zeta \tag{15.3}$$

$$\delta\theta_{\mu\nu} = \tfrac{1}{4}\mathrm{i}\bar{\zeta}(\sigma_{\mu\rho}\partial^\rho J_\nu + \sigma_{\nu\rho}\partial^\rho J_\mu),$$

where X, Y are a scalar–pseudoscalar pair, $j_\mu^{(5)}$ is an axial vector, J_μ is a Majorana spinor-vector and $\theta_{\mu\nu}$ is a symmetric tensor. In our particular case of the Wess–Zumino model, these fields are given in terms of the matter fields by

$$X = \frac{m}{3}(A^2 - B^2), \qquad Y = \frac{2m}{3}AB$$

$$j_\mu^{(5)} = \tfrac{2}{3}A\overset{\leftrightarrow}{\partial}_\mu B + \tfrac{1}{6}\mathrm{i}\bar{\psi}\gamma_\mu\gamma_5\psi$$

$$J_\mu = \slashed{\partial}(A - \gamma_5 B)\gamma_\mu\psi + \mathrm{i}m\gamma_\mu(A - \gamma_5 B)\psi + \mathrm{i}g\gamma_\mu(A - \gamma_5 B)^2\psi - \frac{2\mathrm{i}}{3}\partial^\nu[\sigma_{\mu\nu}(A + \gamma_5 B)\psi] \tag{15.4}$$

$$\theta_{\mu\nu} = \partial_\mu A\,\partial_\nu A - \tfrac{1}{6}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)A^2 + \partial_\mu B\,\partial_\nu B - \tfrac{1}{6}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)B^2 + \tfrac{1}{4}\mathrm{i}\bar{\psi}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\psi - \eta_{\mu\nu}L,$$

with the L in $\theta_{\mu\nu}$ being the Wess–Zumino Lagrangian (5.4).

Quite independently of the particular realization (15.4), the fields of (15.3) form a representation of the supersymmetry algebra, provided only that the conservation law holds and that the stress tensor is symmetric:

$$\partial^\mu J_\mu = \partial^\mu\theta_{\mu\nu} = 0 \tag{15.5}$$

$$\theta_{\mu\nu} = \theta_{\nu\mu}. \tag{15.6}$$

Both properties are preserved by the transformation laws and are guaranteed for the particular realization (15.4) in terms of on-shell matter fields.

15.2. The "multiplet of anomalies"

The supercurrent multiplet (15.3) is not irreducible. The fields X and Y, the γ-trace of J_μ, the trace of $\theta_{\mu\nu}$ and the divergence of $j_\mu^{(5)}$ transform into each other and thus form a submultiplet. Closer study will reveal that this is a chiral multiplet with components

$$\phi = (X, Y; -\tfrac{1}{3}\mathrm{i}\gamma^\mu J_\mu; \tfrac{2}{3}\theta_\mu{}^\mu, \partial^\mu j_\mu^{(5)}) \tag{15.7}$$

in the notation of eqs. (4.4) and (4.5). In the particular realization (15.4), all component fields of this multiplet will disappear when $m = 0$. This is, of course, only a classical result: the trace of the stress tensor as well as the divergence of a chiral current are expected to receive *anomalous contributions* even for a massless model. Provided that the supersymmetry current itself has no anomalies, i.e. that

$\partial^{\mu} J_{\mu} = 0$ remains true, these anomalies must sit in a multiplet of supersymmetry, a "multiplet of anomalies", of which the chiral multiplet (15.7) is an example. Being part of the same irreducible multiplet as $\theta_{\mu}{}^{\mu}$ and $\partial^{\mu} j_{\mu}^{(5)}$, the γ-trace of the supercurrent will get anomalous contributions as well. The "nice-ness" condition of the previous subsection was nothing but the demand that the multiplet of anomalies vanish for zero mass at the classical level.

In the case of

$$\phi = 0 , \tag{15.8}$$

the multiplet (15.3) reduces to a smaller, irreducible multiplet, the *conformal current multiplet*

$$
\begin{aligned}
\delta j_{\mu}^{(5)} &= -\bar{\zeta}\gamma_5 J_{\mu} \\
\delta J_{\mu} &= -2i\gamma^{\nu}\zeta\theta_{\mu\nu} + i\gamma^{\nu}\gamma_5\zeta\partial_{\nu}j_{\mu}^{(5)} + \tfrac{1}{2}i\varepsilon_{\mu}{}^{\nu\rho\sigma}\gamma_{\nu}\zeta\partial_{\rho}j_{\sigma}^{(5)} \\
\delta\theta_{\mu\nu} &= \tfrac{1}{4}i\bar{\zeta}(\sigma_{\mu\rho}\partial^{\rho}J_{\nu} + \sigma_{\nu\rho}\partial^{\rho}J_{\mu})
\end{aligned}
\tag{15.9}
$$

which represents supersymmetry if all quantities are conserved, if J_{μ} is γ-traceless and if $\theta_{\mu\nu}$ is symmetric and traceless:

$$0 = \partial^{\mu} j_{\mu}^{(5)} = \partial^{\mu} J_{\mu} = \partial^{\mu}\theta_{\mu\nu} \tag{15.10}$$

$$0 = \gamma^{\mu} J_{\mu} = \theta_{\mu\nu} - \theta_{\nu\mu} = \theta_{\mu}{}^{\mu} . \tag{15.11}$$

As suggested by the absence of a trace in the stress tensor, the multiplet (15.9) with conditions (15.10) and (15.11) is closely related to *conformal symmetry*. The multiplet of conformal currents is actually unique [68], but there are many ways of introducing a trace for $\theta_{\mu\nu}$ other than by a chiral anomalies multiplet [65, 66].

15.3. The charges

The conservation laws for J_{μ} and $\theta_{\mu\nu}$ allow to define time-independent charge operators as integrals of J_0 and $\theta_{0\mu}$ over 3-space, eqs. (15.1) and (15.2). The transformation laws (15.3) allow to actually calculate the algebra for these charges. Thus we get, as expected,

$$
\begin{aligned}
[Q, \bar{Q}\zeta] &= i\int d^3x\, \delta J_0 \\
&= 2\gamma^{\mu}\zeta\int d^3x\,\theta_{0\mu} - \gamma^{\mu}\gamma_5\zeta\int d^3x\,(\partial_{\mu}j_0^{(5)} - \eta_{0\mu}\partial^{\nu}j_{\nu}^{(5)}) \\
&\quad - \tfrac{1}{2}\varepsilon_0{}^{\mu\rho\sigma}\gamma_{\mu}\zeta\int d^3x\,\partial_{\rho}j_{\sigma}^{(5)} + \sigma_{0\mu}\zeta\int d^3x\,\partial^{\mu}X + \sigma_{0\mu}\gamma_5\zeta\int d^3x\,\partial^{\mu}Y \\
&= 2\gamma^{\mu}\zeta P_{\mu} .
\end{aligned}
$$

The terms with $j_{\mu}^{(5)}$, X and Y all vanished because we assumed that surface terms are zero,

$$\int d^3x \, \partial^i (\text{anything}) = 0 \qquad \text{for } i = 1, 2, 3 .$$ (15.12)

Thus the term $\partial_\mu j_0^{(5)} - \eta_{0\mu} \partial^\nu j_\nu^{(5)}$ can only contribute for $\mu = \nu = 0$ when it vanishes anyway. The other terms involve only spatial derivatives in the first place. In a similar fashion, it can be shown from $\delta\theta_{\mu\nu}$ that $[Q, P] = 0$.

The symmetry of $\theta_{\mu\nu}$ allows the construction of a further conserved current as a first moment of $\theta_{\mu\nu}$:

$$m_{\mu\nu\rho} \equiv x_\mu \theta_{\nu\rho} - x_\nu \theta_{\mu\rho} \qquad \text{with} \qquad \partial^\rho m_{\mu\nu\rho} = 0 .$$ (15.13)

If we define

$$M_{\mu\nu} \equiv \int d^3x \, m_{\mu\nu 0} ,$$ (15.14)

we find

$$[M_{\mu\nu}, \bar\zeta Q] = i \int d^3x \, (x_\mu \, \delta\theta_{\nu 0} - x_\nu \, \delta\theta_{\mu 0})$$

$$= -\tfrac{1}{4}\bar\zeta \int d^3x \, [x_\mu \sigma_{\nu\rho} \, \partial^\rho J_0 + x_\mu \sigma_{0\rho} \, \partial^\rho J_\nu - (\mu \leftrightarrow \nu)] .$$

To evaluate this requires some algebra. Using the conservation law for J_μ, we first write

$$x_\mu \sigma_{\nu\rho} \, \partial^\rho J_0 = x_\mu \sigma_{\nu 0} \, \partial^0 J_0 + x_\mu \sigma_{\nu i} \, \partial^i J_0 = x_\mu \, \partial^i (\sigma_{\nu i} J_0 - \sigma_{\nu 0} J_i)$$

$$x_\mu \sigma_{0\rho} \, \partial^\rho J_\nu = x_\mu \sigma_{0i} \, \partial^i J_\nu .$$

The spatial derivatives now do contribute since they act on the explicit factors of x when partially integrated, and we get

$$[M_{\mu\nu}, \bar\zeta Q] = -\tfrac{1}{4}\bar\zeta \int d^3x \, \eta_\mu{}^i \, (\sigma_{\nu 0} J_i - \sigma_{\nu i} J_0 - \sigma_{0i} J_\nu) - (\mu \leftrightarrow \nu)$$

$$= -\tfrac{1}{4}\bar\zeta \int d^3x \, (\sigma_{\nu 0} J_\mu - \sigma_{\nu\mu} J_0 - \sigma_{0\mu} J_\nu) - (\mu \leftrightarrow \nu)$$

$$= -\tfrac{1}{2}\bar\zeta \sigma_{\mu\nu} Q ,$$

exactly as expected from eq. (4.2).

In the case of the multiplet of currents (15.3), Q, P_μ and $M_{\mu\nu}$ are all the charges that can be constructed. They are just the symmetries of $N = 1$ Poincaré supersymmetry. In particular, there is no chiral R-charge. This is readily understandable in terms of the currents, as there is no conserved axial-vector. For the massive Wess–Zumino model itself, we can derive the absence of R-symmetry directly: according to eqs. (7.27), (7.28) and (7.45), the F-component of a superfield will only be

R-invariant if the superfield has R-weight $n = -2$. Thus, the mass term $m(\phi^2)_F$ and the interaction term $g(\phi^3)_F$ can never be simultaneously R-invariant because ϕ^2 and ϕ^3 cannot both have $n = -2$.

In the absence of the mass term, however, it is the shorter current multiplet (15.9) that is associated with the model, at least at the classical level. This does have a conserved chiral current and a corresponding charge

$$R \equiv \int \mathrm{d}^3 x\, j_0^{(5)} \tag{15.15a}$$

with

$$[R, \bar\zeta Q] = \mathrm{i} \int \mathrm{d}^3 x\, \delta j_0^{(5)} = -\mathrm{i}\bar\zeta\gamma_5 Q, \tag{15.15b}$$

as required by (4.2). The explicit form of $j_\mu^{(5)}$ gives, with the canonical quantisation rules for the matter fields,

$$[A + \mathrm{i}B, R] = \tfrac{2}{3} \int \mathrm{d}^3 x\, [A + \mathrm{i}B, A\dot B - \dot A B + \tfrac{1}{4}\mathrm{i}\bar\psi\gamma_0\gamma_5\psi] = \tfrac{2}{3}(A + \mathrm{i}B)$$

which corresponds to a chiral weight of $n = -\tfrac{2}{3}$ for the matter superfield, see eqs. (7.27) and (7.28). The superfield ϕ^3 then has $n = -2$, as required for the invariance of the interaction term.

Apart from R, the conformal current multiplet gives rise to further charges because the tracelessness conditions in (15.11) allow the construction of further conserved moments, namely

$$d_\mu = x^\nu \theta_{\mu\nu} \qquad \text{with } \partial^\mu d_\mu = 0$$

$$k_{\mu\rho} = 2x_\mu x^\nu \theta_{\nu\rho} - x^2 \theta_{\mu\rho} \qquad \text{with } \partial^\rho k_{\mu\rho} = 0 \tag{15.16}$$

$$s_\mu = \mathrm{i}x^\nu \gamma_\nu J_\mu \qquad \text{with } \partial^\mu s_\mu = 0.$$

The new charges are

$$D = \int \mathrm{d}^3 x\, d_0$$

$$K_\mu = \int \mathrm{d}^3 x\, k_{\mu 0} \tag{15.17}$$

$$S_\alpha = \int \mathrm{d}^3 x\, s_{0\alpha},$$

and their algebraic properties can be evaluated in a manner very similar to the example of $[Q, M_{\mu\nu}]$ given above. The result will be the *algebra of $N = 1$ conformal supersymmetry*, to be discussed in more detail in the next section.

15.4. The supercurrent superfield

We now turn our attention to the superspace formulation of the current multiplet. In order to derive a superfield form for the multiplet (15.3), we first study its representation in terms of matter fields (15.4) and translate that into superfield language. We find

$$X + \mathrm{i}\, Y = \tfrac{1}{3} m\, [\phi^2]_{\theta = \bar{\theta} = 0}$$

$$\sigma^\mu_{\alpha\dot\alpha}\, j^{(5)}_\mu = \tfrac{1}{3}\mathrm{i}\, [\phi \overleftrightarrow{\partial}_{\alpha\dot\alpha} \bar\phi]_{\theta = \bar{\theta} = 0} - \tfrac{1}{6} [D_\alpha \phi\, \bar{D}_{\dot\alpha} \bar\phi]_{\theta = \bar{\theta} = 0} \tag{15.18}$$

and the second equation suggests the definition of a real superfield

$$V_{\alpha\dot\alpha} \equiv D_\alpha \phi\, \bar{D}_{\dot\alpha} \bar\phi - 2\mathrm{i}\phi \overleftrightarrow{\partial}_{\alpha\dot\alpha} \bar\phi, \tag{15.19}$$

so that $\sigma^\mu_{\alpha\dot\alpha}\, j^{(5)}_\mu = -\tfrac{1}{6} (V_{\alpha\dot\alpha})_{\theta = \bar{\theta} = 0}$.

The components $X + \mathrm{i}\, Y$ and $j^{(5)}_\mu$ have the same dimension so that each could be the lowest component of unrestricted superfields. The transformation laws for $j^{(5)}_\mu$, X and Y, however, are related by the fact that $\gamma^\mu J_\mu$ appears in all of them. This will constrain the two superfields, and we now proceed to determine the exact nature of that constraint.

A little bit of experience teaches that the chiral projection of $\gamma^\mu J_\mu$ will be proportional to the lowest component of the superfield $\bar{D}^{\dot\alpha} V_{\alpha\dot\alpha}$. We evaluate, from (15.19) and using the chirality condition $\bar{D}\phi = 0$,

$$\bar{D}^{\dot\alpha} V_{\alpha\dot\alpha} = -\bar{D}_{\dot\alpha} D_\alpha \phi\, \bar{D}^{\dot\alpha} \bar\phi + D_\alpha \phi\, \bar{D}^2 \bar\phi - 2\mathrm{i}\phi \overleftrightarrow{\partial}_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} \bar\phi.$$

This can be further evaluated, again using the chirality condition:

$$-\bar{D}_{\dot\alpha} D_\alpha \phi\, \bar{D}^{\dot\alpha} \bar\phi = -\{D_\alpha, \bar{D}_{\dot\alpha}\}\phi\, \bar{D}^{\dot\alpha} \bar\phi = -2\mathrm{i}\, \partial_{\alpha\dot\alpha} \phi\, \bar{D}^{\dot\alpha} \bar\phi$$

$$-2\mathrm{i}\phi \overleftrightarrow{\partial}_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} \bar\phi = 2\mathrm{i}\, \partial_{\alpha\dot\alpha} \phi\, \bar{D}^{\dot\alpha} \bar\phi - \mathrm{i}\phi\, \partial_{\alpha\dot\alpha}\, \bar{D}^{\dot\alpha} \bar\phi - \tfrac{1}{2} \bar{D}^{\dot\alpha}\{D_\alpha, \bar{D}_{\dot\alpha}\}\bar\phi.$$

The last term in this is

$$-\tfrac{1}{2}\phi\, \bar{D}^{\dot\alpha}\{D_\alpha, \bar{D}_{\dot\alpha}\}\bar\phi = \tfrac{1}{2}\phi\, \bar{D}_{\dot\alpha} D_\alpha \bar{D}^{\dot\alpha} \bar\phi = \tfrac{1}{2}\phi\, \{\bar{D}_{\dot\alpha}, D_\alpha\}\bar{D}^{\dot\alpha} \bar\phi - \tfrac{1}{2}\phi\, D_\alpha \bar{D}^2 \bar\phi$$

$$= \mathrm{i}\phi\, \partial_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} \bar\phi - \tfrac{1}{2}\phi\, D_\alpha \bar{D}^2 \bar\phi.$$

Collecting terms, we find

$$\bar{D}^{\dot\alpha} V_{\alpha\dot\alpha} = D_\alpha \phi\, \bar{D}^2 \bar\phi - \tfrac{1}{2}\phi\, D_\alpha \bar{D}^2 \bar\phi.$$

We now use the equation of motion (8.20) and find very straightforwardly

$$\bar{D}^{\dot\alpha} V_{\alpha\dot\alpha} = m\, D_\alpha \phi^2. \tag{15.20}$$

If we evaluate the θ-$\bar{\theta}$-independent component of this, we find agreement with the properties of $\delta j_\mu^{(5)}$, δX and δY.

Let us, as abstractly as possible, consider the properties of eq. (15.20). In the absence of a mass term, the right-hand side will be zero,

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} = 0 . \tag{15.21}$$

We can evaluate this condition without recourse to any realization of $V_{\alpha\dot{\alpha}}$ in terms of matter fields and find that (15.21) is completely sufficient to define the *conformal current superfield* with component transformation laws as in eqs. (15.9). In the presence of a chiral anomalies multiplet,

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} = D_\alpha S \qquad \text{with} \qquad \bar{D}_{\dot{\alpha}} S = 0 , \tag{15.22}$$

we get instead the *Poincaré supercurrent superfield* with component transformation laws as in eqs. (5.3).

In the case of the gauge multiplet, the supercurrent superfield is given by

$$V_{\alpha\dot{\alpha}} = -12 \, \text{tr} \, W_\alpha \bar{W}_{\dot{\alpha}} . \tag{15.23}$$

The property (15.21) then follows from the constraints $0 = \nabla^\alpha W_\alpha + \bar{\nabla}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} = \bar{\nabla}_{\dot{\alpha}} W_\alpha$ and the equation of motion $\nabla^\alpha W_\alpha - \bar{\nabla}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} = 0$.

16. Conformal supersymmetry

In section 15, we saw that the multiplet of currents is particularly simple and short if no masses are present in an $N = 1$ supersymmetric model, and if we allow ourselves to ignore possible anomalies. The properties of the currents are such that we can construct conserved charges R, P_μ, $M_{\mu\nu}$, D, K_μ, Q_α and S_α from them. In the present section, we shall explore the algebra of these charges [71], generalise the results to $N > 1$ [11, 34], and give representations for the $N = 1$ *superconformal algebra* on fields [71, 58].

16.1. Deriving the N = 1 algebra from the currents

Assume the only conserved current to be a symmetric energy-momentum tensor $\theta_{\mu\nu}$ with non-zero trace. The only charges are P_μ and $M_{\mu\nu}$, defined by (15.2) and (15.13–14). Assume further that the action of P_μ on fields is given by the derivative, eq. (3.42b), and that surface terms vanish in the d^3x-integrals, eq. (15.12). Then we can show that

$$[P_\mu, P_\nu] = \int d^3x \, [\theta_{0\mu}, P_\nu] = i \int d^3x \, \partial_\nu \theta_{0\mu} = 0$$

$$[M_{\mu\nu}, P_\rho] = i \int d^3x \, (x_\mu \, \partial_\rho \theta_{\nu 0} - x_\nu \, \partial_\rho \theta_{\mu 0}) = i\eta_{\nu\rho} P_\mu - i\eta_{\mu\rho} P_\nu .$$

The intermediate steps are very similar to those displayed after eq. (15.14). The $[M, M]$-commutator follows from the Jacobi identities, and we have rederived the algebra (2.18) of the Poincaré group.

Next we assume $\theta_{\mu\nu}$ to be traceless. This introduces the new charges D and K_μ, viz. (15.16) and (15.17). Lorentz covariance requires D to be a scalar and K_μ to be a vector. By explicit calculation we find

$$[P_\mu, D] = -i \int d^3x\, x^\nu\, \partial_\mu \theta_{0\nu} = iP_\mu\,,$$

again with intermediate steps as before. Knowing that no further charges can be derived from $\theta_{\mu\nu}$, we can postulate that the algebra must close on the 15 charges P_μ, $M_{\mu\nu}$, D and K_μ. Surprisingly enough, this closure requirement together with what we already know about the algebra is sufficient to determine the remaining commutators $[P, K]$, $[K, D]$ and $[K, K]$ from the Jacobi identities alone, without further recourse to the currents. The result is the *algebra of the conformal group*

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho})$$

$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho)$$

$$[K_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho)$$

$$[D, M_{\mu\nu}] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0 \qquad\qquad (16.1)$$

$$[P_\mu, D] = iP_\mu\,; \qquad [K_\mu, D] = -iK_\mu$$

$$[P_\mu, K_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu})\,.$$

There is a rescaling freedom for K_μ which would change the factor of 2 in the last equation; as it stands, the normalisation is compatible with our definition of K_μ from the current. A vanishing $[P, K]$ commutator would also be compatible with the Jacobi identities, in that case there would be an arbitrary real factor in $[K, D]$. With this exception, the algebra of the conformal group follows from the rather weak assumptions made above.

Next we include into our analysis the conserved, γ-traceless Majorana supercurrent J_μ. We have already shown in subsection 15.3 that the charge Q, defined by (15.1) from J_μ, completes the algebra of the Poincaré group to that of $N = 1$ Poincaré supersymmetry. The relationship of supersymmetry with the conformal algebra (16.1) is completely defined by the commutator $[Q, K_\mu]$, which after some calculation is found to be

$$[\bar\zeta Q, K_\mu] = -i \int d^3x\, (2x_\mu x^\nu\, \delta\theta_{\nu 0} - x^2\, \delta\theta_{\mu 0}) = \bar\zeta\gamma_\mu S\,,$$

with S the charge defined in (15.17) from the first moment s_μ of J_μ. The commutator $[Q, K_\mu]$ does not have a γ-traceless part. This property can be written as

$$[Q, K_\mu] = \tfrac{1}{4}\gamma_\mu\gamma^\nu[Q, K_\nu] = \gamma_\mu S\,, \qquad\qquad (16.2)$$

and is actually sufficient to determine all relationships of the *algebra of conformal supersymmetry*, once the Poincaré superalgebra and the conformal algebra are given. The calculation can be done solely by means of Jacobi identities, without use of the current multiplet. It leads to the following algebra

relationships beyond those of (16.1):

$$[Q, M_{\mu\nu}] = \tfrac{1}{2}\sigma_{\mu\nu}Q; \qquad [S, M_{\mu\nu}] = \tfrac{1}{2}\sigma_{\mu\nu}S$$

$$[Q, D] = \tfrac{1}{2}iQ; \qquad [S, D] = -\tfrac{1}{2}iS$$

$$[Q, P_{\mu}] = 0; \qquad [S, P_{\mu}] = \gamma_{\mu}Q$$

$$[Q, K_{\mu}] = \gamma_{\mu}S; \qquad [S, K_{\mu}] = 0 \qquad\qquad (16.3)$$

$$[Q, R] = i\gamma_5 Q; \qquad [S, R] = -i\gamma_5 S$$

$$[R, M_{\mu\nu}] = [R, P_{\mu}] = [R, D] = [R, K_{\mu}] = 0$$

$$\{Q, \bar{Q}\} = 2\gamma^{\mu}P_{\mu}; \qquad \{S, \bar{S}\} = 2\gamma^{\mu}K_{\mu}$$

$$\{S, \bar{Q}\} = 2iD + \sigma^{\mu\nu}M_{\mu\nu} + 3i\gamma_5 R.$$

The S is defined in (16.2) as a certain multiple of the γ-trace of $[Q, K]$. The normalisation here is compatible with the definition in (15.17). Similarly, the charge R can be seen as *defined* by the last equation in (16.3). Its commutators with the other charges are fixed by Jacobi identities and we cannot set $R = 0$, since the $[Q, R]$ commutator is not zero, but we could arbitrarily rescale R. The chosen scale is the one compatible with the alternative definition via the chiral current, eq. (15.15).

16.2. More about the conformal group

Four-dimensional Minkowski space is a coset space G/H not only of the Poincaré group but also of the conformal group, with H the subgroup generated by $M_{\mu\nu}$, D and K_{μ}. We can therefore use the techniques of section 7 to determine the action of conformal transformations on Minkowski space [43]. The coset can be parametrized as in eq. (7.13) and we get the same relization for Lorentz rotations and translations as before, eqs. (7.14). For infinitesimal D-transformations (*dilatations*) and K-transformations (*special conformal transformations*) we get

$$(1 + i\varepsilon D)\, e^{ix\cdot P} = e^{ix\cdot P}\, e^{-ix\cdot P}(1 + i\varepsilon D)\, e^{ix\cdot P}$$

$$= e^{i(1+\varepsilon)x\cdot P}(1 + i\varepsilon D) + o(\varepsilon^2)$$

$$(1 + ia^{\mu}K_{\mu})\, e^{ix\cdot P} = \exp\{i(x^{\mu} + 2x\cdot ax^{\mu} - x^2 a^{\mu})P_{\mu}\}(1 + ia^{\mu}K_{\mu} + 2ix\cdot aD - 2ix^{\mu}a^{\nu}M_{\mu\nu}) + o(a^2).$$

The equivalents of eqs. (3.42) and (7.8), namely the infinitesimal version of the action of the generators on fields over Minkowski space, are thus

$$[\phi, P_{\mu}] \;\; = i\, \partial_{\mu}\phi$$

$$[\phi, M_{\mu\nu}] = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\phi + \Sigma_{\mu\nu}\phi$$

$$[\phi, D] \;\;\; = ix^{\mu}\, \partial_{\mu}\phi + i\Delta\phi \qquad\qquad (16.4)$$

$$[\phi, K_{\mu}] \;\; = i(2x_{\mu}x\cdot\partial - x^2\, \partial_{\mu})\phi + (\kappa_{\mu} + 2ix_{\mu}\Delta + 2x^{\nu}\Sigma_{\mu\nu})\phi.$$

The matrices $\Sigma_{\mu\nu}$, $i\Delta$ and κ_{μ} act on indices of ϕ and are a finite-dimensional representation of the

algebra of the "little group" H. Since $[\Delta, \Sigma_{\mu\nu}] = 0$, we know from Schur's lemma that Δ must be a number if the $\Sigma_{\mu\nu}$ generate an irreducible representation of the Lorentz group. This number is the *dimension* of the field ϕ. This dimension has already come up on several occasions, first in subsection 4.7.

Whereas the coordinate transformations associated with the Poincaré group leave the line element

$$ds^2 \equiv \eta_{\mu\nu} \, dx^\mu \, dx^\nu \tag{16.5}$$

invariant, the dilatations and special conformal transformations *rescale* it in an x-dependent way,

$$ds^2 \to \sigma(x) \, ds^2 . \tag{16.6}$$

Angles between lines are preserved, hence the name "conformal". The 15-parameter conformal group which we encounter here is actually the most general connected group of coordinate transformations in four dimensions with property (16.6).

The conformal group is the pseudo-orthogonal group O(2, 4). This is easily seen by defining additional "Lorentz" generators

$$M_{\mu 5} \equiv \tfrac{1}{2}(P_\mu - K_\mu); \qquad M_{\mu 6} \equiv \tfrac{1}{2}(P_\mu + K_\mu); \qquad M_{56} \equiv -D \tag{16.7}$$

and arranging them together with the $M_{\mu\nu}$ into an antisymmetric 6×6 matrix M_{ab}. The commutation relations (16.1) can then be rewritten in a single equation of the form typical for the algebra of an orthogonal group:

$$[M_{ab}, M_{cd}] = i(\eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc}) . \tag{16.8}$$

The metric elements η_{ab} are the usual $\eta_{\mu\nu}$, augmented by the new non-zero elements

$$-\eta_{55} = \eta_{66} = +1 , \tag{16.9}$$

which make the metric that of O(2, 4).

A spinor representation of this algebra is easily found, after all the groundwork laid in appendix A.7 and section 14. The O(2, 4) Dirac matrices are those of (14.14) but with Γ_6 multiplied with i to account for the difference between O(1, 5), the six-dimensional Lorentz group, and O(2, 4), the four-dimensional conformal group. We then take the representation $\tfrac{1}{2}\Sigma_{ab}$, defined in (14.2) and (14.5), and call it $\tfrac{1}{2}\sigma_{ab}$. We find that the σ_{ab} are the standard 4×4 matrices $\sigma_{\mu\nu}$, augmented by

$$\sigma_{\mu 5} = i\gamma_\mu \gamma_5; \qquad \sigma_{\mu 6} = \gamma_\mu ; \qquad \sigma_{56} = \gamma_5 . \tag{16.10}$$

Since all of these are traceless and have the property $\gamma_0(\sigma_{ab})^\dagger \gamma_0^{-1} = \sigma_{ab}$, with two eigenvalues $+1$ and two eigenvalues -1 for γ_0, they generate SU(2, 2), the *covering group of* O(2, 4).

16.3. Conformal spinors

This spinor representation of the algebra O(2, 4) is four-dimensional and complex. In the language of appendix A.7, it is a representation with a chirality condition. We could also have Majorana spinors

with eight real components, but we cannot have spinors that are both Majorana and chiral since the entry in table A.4 for $d = 6$ and $d_- = 4$ is $+-$ and not $++$. At least eight real components are thus required for a conformal spinor, which explains the necessity for the additional charges S_α in the $N = 1$ superconformal algebra.

A conformal spinor can be constructed from the chiral components of the Lorentz spinors Q and S in the following way [16]:

$$\Sigma = \begin{bmatrix} Q_\alpha \\ \bar{S}^{\dot\alpha} \end{bmatrix}.$$
(16.11)

Like always when a covariant quantity has been properly defined, the algebra (16.3) simplifies enormously and can actually be written in three lines, using the Σ's:

$$[\Sigma, M_{ab}] = \tfrac{1}{2}\sigma_{ab}\Sigma; \qquad [\bar{\Sigma}, M_{ab}] = -\tfrac{1}{2}\bar{\Sigma}\sigma_{ab}$$

$$[\Sigma, R] = \Sigma; \qquad [\bar{\Sigma}, R] = -\bar{\Sigma}; \qquad [M_{ab}, R] = 0$$
(16.12)

$$\{\Sigma, \Sigma\} = \{\bar{\Sigma}, \bar{\Sigma}\} = 0; \qquad \{\Sigma, \bar{\Sigma}\} = \sigma^{ab}M_{ab} - 3R.$$

The constraint (16.2) can now be rephrased in terms of representations of SU(2, 2): there are no charges present which are in the representations **20** or **36** of SU(2, 2). Hence the commutator of Σ with M_{ab} which in general could decompose as $4 \otimes 15 = 4 \oplus 20^* \oplus 36$ only contains a **4**, namely the Σ's themselves. This property connects constraint (16.2) with the arguments of ref. [34] and of section 2, because the anticommutator of a **20** with the **20*** or of a **36** with the **36*** would necessarily contain non-trivial tensor representations of SU(2, 2) other than the **15** and thus must be zero (this is an extrapolation of the Coleman–Mundula theorem). The positive metric assumption for the Hilbert space then means that the **20** and the **36** must be absent.

16.4. The algebra of extended conformal supersymmetry

Just as extended Poincaré supersymmetry has several charges $Q_{\alpha i}$, so extended conformal supersymmetry [11, 34, 58] has several Σ_i, $i = 1, \ldots N$. Stretching the Coleman–Mundula theorem somewhat, we can rule out bosonic charges in non-trivial representations of the conformal group other than the conformal generators themselves. This reduces the possible extensions of (16.12) to

$$[\Sigma_i, M_{ab}] = \tfrac{1}{2}\sigma_{ab}\Sigma_i; \qquad [\bar{\Sigma}^i, M_{ab}] = -\tfrac{1}{2}\bar{\Sigma}^i\sigma_{ab}$$

$$\{\Sigma_i, \Sigma_j\} = \{\bar{\Sigma}^i, \bar{\Sigma}^j\} = 0; \qquad \{\Sigma_i, \bar{\Sigma}^j\} = \delta_i^j \sigma^{ab}M_{ab} - 4B_i^j.$$
(16.13)

The last equation defines charges B_i^j as the most general right-hand side compatible with the assumptions about the structure of the bosonic subalgebra and with conformal covariance. The Jacobi identities fix the remaining commutators

$$[B_i^j, M_{ab}] = 0$$

$$[\Sigma_i, B_j^k] = \delta_i^k \Sigma_j - \tfrac{1}{4}\delta_j^k \Sigma_i; \qquad [\bar{\Sigma}^i, B_j^k] = -\delta_j^i \bar{\Sigma}^k + \tfrac{1}{4}\delta_j^k \bar{\Sigma}^i$$
(16.14)

$$[B_i^j, B_k^l] = \delta_i^l B_k^j - \delta_k^j B_i^l,$$

and from the Hermiticity property of the last of eqs. (16.13) we get

$$(B_i{}^j)^\dagger = B_j{}^i .$$

(16.15)

These fix the traceless part of $B_i{}^j$ to be the generators of SU(N). The non-zero commutator of Σ with all traceless combinations of $B_i{}^j$ means that the internal symmetry group must contain the full SU(N), not just a subgroup. For the trace part we define

$$R \equiv \frac{4}{4 - N} B_i{}^i \qquad \text{for } N \neq 4$$

(16.16)

and find

$$[\Sigma_i, R] = \Sigma_i ; \qquad [\bar\Sigma^i, R] = -\bar\Sigma^i .$$

(16.17)

The internal symmetry algebra is thus that of U(N) for $N \neq 4$.

For the special case of $N = 4$, I already mentioned the absence of a chiral scalar charge at the end of subsection 13.1. This now makes sense, since $B_i{}^i$ commutes with all Σ's and may thus be absent in the $\{\Sigma, \bar\Sigma\}$ anticommutator. The algebra would actually allow such a trace term in $\{\Sigma, \bar\Sigma\}$, but it must commute with the Σ's. Elsewhere I have shown [67] that this trace term is absent if and only if the lowest helicity in the representation multiplet is $\lambda_0 = -1$, i.e. for the Yang–Mills multiplet. Conformal $N = 4$ supergravity will have such a term. The algebra would also allow an R-charge for $N = 4$ which is an outer automorphism, i.e. which is not generated by the anticommutators.

16.5. Multiplets of $N = 1$ conformal supersymmetry

Already in Wess and Zumino's first paper [71], the transformations of the chiral multiplet (4.4) and of the general multiplet (4.6) were given not only for Poincaré supersymmetry transformations but also for conformal ones. In the present subsection, these will be rederived in a more general context.

Since the commutator of Q with K_μ gives the conformal spinor charge S, it should be possible to make a superconformal multiplet out of every multiplet of Poincaré supersymmetry by postulating suitable transformation laws of the fields under K-transformations. These will be of the form given in eq. (16.4), and initially there will be some freedom of choice for the "little group" representation κ_μ, $i\Delta$, $\Sigma_{\mu\nu}$. Most importantly, however, there is the constraint (16.2) which must be fulfilled by the Q and the K-transformations, namely that a γ_μ can be "split off" from the commutator $[Q, K_\mu]$. We shall see that this will restrict the possible little group representations for the fields of a superconformal multiplet.

We start by imposing as few restrictions as possible on a *general multiplet* of superconformal symmetry. A (complex) field C is taken as the starting point of the multiplet, just as in subsection 4.2. Poincaré supersymmetry transformations will generate a multiplet

$$V = (C; \chi; M, N, A_\mu; \lambda; D)$$

(16.18)

out of C, with no reality restrictions on any of the fields. Let us assume that C transforms under R-transformations, dilatations and Lorentz transformations as

$$[C, R] = \mathring{n}C$$

$$[C, D] = ix \cdot \partial C + i\mathring{\Delta}C \qquad\qquad\qquad (16.19)$$

$$[C, M_{\mu\nu}] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)C + \mathring{\Sigma}_{\mu\nu}C,$$

and let us impose the restriction that the matrix representation $\mathring{\Sigma}_{\mu\nu}$ of the Lorentz group (this acts on unwritten indices of C) is *irreducible*. Because of $[\mathring{\Sigma}_{\mu\nu}, \Delta] = 0$ and Schur's lemma, this means that Δ must be a number. Since all fields of the multiplet can be derived from (multiple) commutators of C with Q, and since we know that $[Q, D] = \frac{1}{2}iQ$, we can actually calculate the representation $i\Delta$ of D for the whole multiplet. The result is a diagonal matrix which acts on a column vector made out of the components of V as given in (16.18):

$$\Delta = \mathring{\Delta}\mathbb{1} + \mathrm{diag}(0, \tfrac{1}{2}, 1, 1, 1, \tfrac{3}{2}, 2). \qquad\qquad\qquad (16.20)$$

The matrix κ_μ, the representation of the special conformal charge K_μ, must have

$$[\kappa_\mu, \Delta] = -\kappa_\mu. \qquad\qquad\qquad (16.21)$$

This, together with (16.20), restricts κ_μ to the form

$$\kappa_\mu = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \end{pmatrix} \qquad\qquad\qquad (16.22)$$

with non-zero entries only where there are stars. In particular,

$$\kappa_\mu C = \kappa_\mu \chi = 0. \qquad\qquad\qquad (16.23a)$$

Dimensional analysis will show that $\kappa_\mu M$ and $\kappa_\mu N$ should have the same dimension as C. Since we do not allow non-local operators such as $\Box^{-1}$, there is no way of constructing a vector of that dimension from the fields of our multiplet, and we conclude that

$$\kappa_\mu M = \kappa_\mu N = 0. \qquad\qquad\qquad (16.23b)$$

The remaining fields A_μ, λ and D can and will have κ-terms. Since we now know the Δ and the $\Sigma_{\mu\nu}$ for all fields of the multiplet ($\Sigma_{\mu\nu}$ is $\mathring{\Sigma}_{\mu\nu}$ plus the vector representation for A_μ or the spinor representation for χ and λ), and know quite a bit about κ_μ, we can plunge into deriving the S-transformations for the multiplet from (4.7) and (16.4). This is a lengthy calculation which cannot possibly be repeated here. It consists of evaluating

$$[.., [Q, K_\mu]\} = [[.., Q\}, K_\mu] - [[.., K_\mu], Q\}$$

for all fields of the multiplet and of imposing the constraint (16.2) at each stage. This results in the remaining κ-terms

$$\kappa_\mu A_\nu = -\varepsilon_{\mu\nu\rho\sigma}\overset{\circ}{\Sigma}{}^{\rho\sigma}C + 3\mathring{n}\eta_{\mu\nu}C$$

$$\kappa_\mu \lambda = -\tfrac{1}{2}\sigma^{\rho\sigma}\gamma_\mu\overset{\circ}{\Sigma}_{\rho\sigma}\chi - \overset{\circ}{\varDelta}\gamma_\mu\chi - \frac{3i\mathring{n}}{2}\gamma_\mu\gamma_5\chi \qquad (16.23c)$$

$$\kappa_\mu D = -2i\overset{\circ}{\varDelta}\partial_\mu C - 3\mathring{n}A_\mu - 2\overset{\circ}{\Sigma}_{\mu\nu}\partial^\nu C + \varepsilon_{\mu\nu\rho\sigma}\overset{\circ}{\Sigma}{}^{\rho\sigma}A^\nu$$

and the transformation laws for the multiplet under combined transformations

$$\delta V = -i[V, \bar{\zeta}Q + \bar{\varepsilon}S]. \qquad (16.24)$$

We use the convenient abbreviations

$$\eta \equiv \zeta - ix^\mu\gamma_\mu\varepsilon; \qquad \overset{\circ}{X}{}^\pm \equiv \overset{\circ}{\varDelta} - \frac{3i\mathring{n}}{2}\gamma_5 \pm \tfrac{1}{2}\sigma^{\mu\nu}\overset{\circ}{\Sigma}_{\mu\nu} \qquad (16.25)$$

and find

$$\delta C = \bar{\eta}\gamma_5\chi$$

$$\delta\chi = (M + \gamma_5 N)\eta - i\gamma^\mu(A_\mu + \gamma_5\partial_\mu C)\eta + 2\overset{\circ}{X}{}^+\gamma_5\varepsilon C$$

$$\delta M = \bar{\eta}(\lambda - i\slashed{\partial}\chi) + \bar{\varepsilon}\overset{\circ}{X}{}^-\chi - 2\bar{\varepsilon}\chi$$

$$\delta N = \bar{\eta}\gamma_5(\lambda - i\slashed{\partial}\chi) - \bar{\varepsilon}\gamma_5\overset{\circ}{X}{}^-\chi + 2\bar{\varepsilon}\gamma_5\chi \qquad (16.26)$$

$$\delta A_\mu = i\bar{\eta}\gamma_\mu\lambda + \partial_\mu(\bar{\eta}\chi) + i\bar{\varepsilon}\overset{\circ}{X}{}^-\gamma_\mu\chi$$

$$\delta\lambda = -i\sigma^{\mu\nu}\eta\partial_\mu A_\nu - \gamma_5\eta D - \overset{\circ}{X}{}^+(M - \gamma_5 N)\varepsilon + i\gamma^\mu\overset{\circ}{X}{}^+(A_\mu + \gamma_5\partial_\mu C)\varepsilon$$

$$\delta D = -i\bar{\eta}\slashed{\partial}\gamma_5\lambda + 2\bar{\varepsilon}\gamma_5\overset{\circ}{X}{}^-(\lambda - \tfrac{1}{2}i\slashed{\partial}\chi).$$

We can now try to impose reality and chirality conditions on this multiplet. If C is to be real,

$$V = V^\dagger, \qquad (16.27)$$

then we must have

$$\mathring{n} = 0 \quad \text{and} \quad \overset{\circ}{\Sigma}_{\mu\nu} = -(\overset{\circ}{\Sigma}_{\mu\nu})^\dagger, \qquad (16.28)$$

i.e., $\overset{\circ}{\Sigma}_{\mu\nu}$ must be the generator of a real representation of the Lorentz group. The transformation laws (16.26) are then consistent with the usual reality (Majorana) conditions on the fields.

A chirality condition consists of imposing a constraint

$$(1 - i\gamma_5)\chi = 0. \qquad (16.29)$$

This will be consistent only if

$$(1 - i\gamma_5)\overset{\circ}{X}{}^+ = 0 \,,$$

i.e., if

$$\overset{\circ}{n} = -\tfrac{2}{3}\overset{\circ}{\Delta} \qquad \text{and} \qquad \overset{\circ}{\Sigma}_{\mu\nu} = \tfrac{1}{2}i\varepsilon_{\mu\nu\rho\sigma}\overset{\circ}{\Sigma}{}^{\rho\sigma} \,. \tag{16.30}$$

This means that a chiral multiplet must be in a self-dual representation of the Lorentz group (it can only have "undotted indices") and that its chiral weight must be $-\tfrac{2}{3}$ times its dimension. With these conditions, (16.29) is consistent, together with

$$0 = M + iN = A_\mu - i\partial_\mu C = \lambda = D \,,$$

and we are left with the fields of the *chiral multiplet*:

$$\phi = (A, B; \psi; F, G)$$
$$C = A + iB; \qquad -\tfrac{1}{4}i(1 + i\gamma_5)\chi = \psi; \qquad M = -i(F - iG) \,. \tag{16.31}$$

If there are no external Lorentz indices, we can write the transformation laws for the chiral multiplet in the usual form involving Majorana spinors:

$$\delta A = \bar{\eta}\psi$$

$$\delta B = \bar{\eta}\gamma_5\psi$$

$$\delta\psi = -(F + \gamma_5 G)\eta - i\slashed{\partial}(A + \gamma_5 B)\eta - 2\overset{\circ}{\Delta}(A - \gamma_5 B)\varepsilon \tag{16.32}$$

$$\delta F = i\bar{\eta}\slashed{\partial}\psi - 2(\overset{\circ}{\Delta} - 1)\bar{\varepsilon}\psi$$

$$\delta G = i\bar{\eta}\gamma_5\slashed{\partial}\psi + 2(\overset{\circ}{\Delta} - 1)\bar{\varepsilon}\gamma_5\psi \,.$$

The canonical value $\overset{\circ}{\Delta} = 1$ is clearly singled out: only then can we start a new chiral multiplet with F and G, the kinetic multiplet, and only then does the chiral weight have the value $-\tfrac{2}{3}$ which is required for invariance of the super-ϕ^3 interaction term, cf. the comments after eqs. (15.15).

Another interesting case is that of

$$\overset{\circ}{X}{}^{\pm} = 0, \qquad \text{i.e.} \qquad \overset{\circ}{\Delta} = \overset{\circ}{n} = \overset{\circ}{\Sigma}_{\mu\nu} = 0 \,, \tag{16.33}$$

when all κ-terms (16.23) vanish and the *curl-submultiplet* is also a submultiplet under conformal transformations.

In the case of

$$\overset{\circ}{X}{}^- = 2, \qquad \text{i.e.} \qquad \overset{\circ}{\Delta} = 2, \qquad \overset{\circ}{n} = \overset{\circ}{\Sigma}_{\mu\nu} = 0 \,, \tag{16.34}$$

we have a chiral submultiplet which starts with M and N. With this set to zero, we get the conformal *linear multiplet* where A_μ must have dimension 3 in order to be conserved.

Let us now turn our attention back to the conditions (16.30) for the existence of a chiral multiplet, and let us try to understand them in a more fundamental way than just by seeing them appear as consequence of a lot of algebra.

In the case of Poincaré supersymmetry, chiral multiplets existed with arbitrary additional Lorentz indices; in other words, the chirality constraint $\bar{D}\phi = 0$ was covariant for arbitrary representations of the "little group", which was the Lorentz group. This is not so for conformal supersymmetry, as could be seen in several ways which involve treatment of conformal supersymmetry in superspace. The simplest way seems to be to consider the shifted superspaces of eq. (7.47). It is clear from (7.48) that a chiral superfield is independent of $\bar{\theta}$ in a 1-type superspace. Indeed, it would have been possible from the beginning to use a superspace G/H with H generated not only by $M_{\mu\nu}$ but also by $\bar{Q}_{\dot{\alpha}}$. This coset is parametrized by x^μ and θ^α alone and is called *chiral superspace*. For a general superfield over chiral superspace, the action of Q would be given by

$$[\phi(x), \bar{Q}_{\dot{\alpha}}\} = 2(\theta\sigma^\mu)_{\dot{\alpha}} \, \partial_\mu\phi + \bar{q}_{\dot{\alpha}}\phi \tag{16.35}$$

and a chiral superfield is one for which the matrix representation for $\bar{Q}$ vanishes:

$$\bar{q}_{\dot{\alpha}} = 0. \tag{16.36}$$

Just as the conformal group could be realized on Minkowski space, so conformal supersymmetry can be realized on Minkowski superspace, with rather complicated x, θ, $\bar{\theta}$ dependent "little group" elements h [58]. It is also possible to realize it on chiral superspace. The "little group" H is then generated by $M_{\mu\nu}$, K_μ, D, S^α, $\bar{S}^{\dot{\alpha}}$ and $\bar{Q}_{\dot{\alpha}}$, and the question whether chiral multiplets exist, can be reduced to the question whether the constraint (16.36) on the representation is consistent. This is non-trivial, because the non-vanishing anti-commutator

$$\{\bar{S}^{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \, M_{\mu\nu} - 2\delta^{\dot{\alpha}}_{\ \dot{\beta}} \, (\tfrac{3}{2}R - \mathrm{i}D),$$

together with the constraint (16.36) implies that the representation of the right-hand side must vanish, i.e., that (16.30) must hold.

Note that it is only required that the Lorentz representation of a chiral superfield be self-dual, not that it vanish. In other words, a chiral conformal superfield can have arbitrarily many undotted but no dotted spinor indices. The chiral weight and the dimension must be related.

We see that these restrictions follow quite directly from the structure of the algebra. Similar restrictions hold for extended conformal supersymmetry [11, 58]. There the Lorentz representation of a chiral superfield must again be self-dual, the SU(N) representation must be trivial, and the R-weight is related to the dimension by

$$n = \frac{2N}{N-4}\Delta. \tag{16.37}$$

17. Concluding remarks

This is the point where one should, in any systematic presentation, begin to talk about supergravity. The current multiplet of section 15 acts as a source for the fields of supergravity in a supersymmetric extension of Einstein's equation. The geometric results of section 10 can be generalised to curved space–time, and the knowledge of conformal supersymmetry from section 16 provides an elegant entry

into supergravity tensor calculus. The technical aspects of supergravity are considerably more complicated than those of flat-space supersymmetry, and I hope to present a second Physics Report in due time where I can give them similar in-depth treatment.

A subject related to flat-space supersymmetry is superGUTs: supersymmetric versions of grand-unified theories. However, since their consistent treatment should not be attempted without reference to the super-Higgs effect, I have omitted them from this report. The super-Higgs effect, arising out of spontaneous breaking of *local* supersymmetry can only be treated in the context of supergravity. Indeed, the absence of massless Goldstone fermions from nature [80, 81] demands that there must be something like a super-Higgs effect, hence that there must be (broken) local supersymmetry, and hence, by virtue of eq. (9.4), there must be (broken) supergravity. Conversely, since curved space does not permit constant spinors (there is no covering group of GL(4) with spinorial representations) we know that the only supersymmetry which can coexist with gravity is local, "gauged" supersymmetry.

The way to proceed from here is to cover the basic concepts of supergravity in a systematic way, and then to apply those to superGUTs and the super-Higgs effect as well as to the other exciting frontiers of present research, super-Kaluza–Klein theories and superstrings. This holds as well for the reader who wishes to learn more about these subjects as for the structure of a second part of this report. Until then, as far as supergravity is concerned, I wish to refer the reader to van Nieuwenhuizen's excellent earlier report [83] which, aimed at a different type of reader, may not present the subject in as much detail as I have done here, but which instead covers a much wider range of developments. I also wish to refer the reader to the excellent books by Wess and Bagger [84] and by Gates, Grisaru, Roček and Siegel [26] which the serious student of our subject will want to read in parallel with and as complements to the present article. (In particular, ref. [26] fills a major gap in my treatment, being entirely devoted to superspace perturbation techniques.)

A. Technical appendix

A.1. Minkowski metric

The flat-space *metric* is taken to be

$$\eta_{\mu\nu} = \mathrm{diag}(+, -, -, -),$$ (A.1)

and the totally *antisymmetric tensor* is normalised such that

$$\varepsilon_{0123} = 1.$$ (A.2)

A.2. Group representations

A matrix *representation* of a group is a homomorphism of the group into a matrix space:

$$r: G \to M \qquad \text{with} \qquad r(g_1) r(g_2) = r(g_1 \circ g_2),$$

where $\circ$ denotes the group product. If $r(g)$ is a homomorphism, then so are $r^*(g)$, $r^{-1T}(g)$ and $r^{-1\dagger}(g)$. We denote the vectors on which these representations act by u_α, $u_{\dot\alpha}$, u^α and $u^{\dot\alpha}$, respectively. The

group transformations on the vectors are then

$$u'_\alpha = r_\alpha{}^\beta u_\beta$$

$$u'_{\dot\alpha} = (r^*)_{\dot\alpha}{}^{\dot\beta} u_{\dot\beta} = u_{\dot\beta} (r^\dagger)^{\dot\beta}{}_{\dot\alpha}$$

$$u'^\alpha = (r^{-1T})^\alpha{}_\beta u^\beta = u^\beta (r^{-1})_\beta{}^\alpha$$

$$u'^{\dot\alpha} = (r^{-1\dagger})^{\dot\alpha}{}_{\dot\beta} u^{\dot\beta} = u^{\dot\beta} (r^{-1*})_{\dot\beta}{}^{\dot\alpha} .$$

(A.3)

If the group is a Lie group, we have infinitesimal group elements for which the representations take the form

$$r_\alpha{}^\beta \approx (1 + i\varepsilon\Sigma)_\alpha{}^\beta; \qquad \text{generator: } \Sigma$$

$$(r^*)_{\dot\alpha}{}^{\dot\beta} \approx (1 - i\varepsilon\Sigma^*)_{\dot\alpha}{}^{\dot\beta}; \qquad \text{generator: } -\Sigma^*$$

$$(r^{-1T})^\alpha{}_\beta \approx (1 - i\varepsilon\Sigma^T)^\alpha{}_\beta; \qquad \text{generator: } -\Sigma^T$$

$$(r^{-1\dagger})^{\dot\alpha}{}_{\dot\beta} \approx (1 + i\varepsilon\Sigma^\dagger)^{\dot\alpha}{}_{\dot\beta}; \qquad \text{generator: } \Sigma^\dagger,$$

(A.4)

and the infinitesimal *variation* of a vector becomes

$$\delta u_\alpha \equiv u'_\alpha - u_\alpha = i\varepsilon\Sigma_\alpha{}^\beta u_\beta ,$$

(A.5)

and similarly for $u_{\dot\alpha}$, u^α and $u^{\dot\alpha}$. The variations of tensors are defined accordingly, e.g.:

$$\delta t_{\alpha\beta} = i\varepsilon\Sigma_\alpha{}^\gamma t_{\gamma\beta} + i\varepsilon\Sigma_\beta{}^\gamma t_{\alpha\gamma}$$

$$\delta t_{\alpha\dot\beta} = i\varepsilon\Sigma_\alpha{}^\gamma t_{\gamma\dot\beta} - i\varepsilon(\Sigma^*)_{\dot\beta}{}^{\dot\gamma} t_{\alpha\dot\gamma} = i\varepsilon\Sigma_\alpha{}^\gamma t_{\gamma\dot\beta} - i\varepsilon t_{\alpha\dot\gamma}(\Sigma^\dagger)^{\dot\gamma}{}_{\dot\beta}$$

(A.6)

$$\delta t_\alpha{}^\beta = i\varepsilon\Sigma_\alpha{}^\gamma t_\gamma{}^\beta - i\varepsilon(\Sigma^T)^\beta{}_\gamma t_\alpha{}^\gamma = i\varepsilon\Sigma_\alpha{}^\gamma t_\gamma{}^\beta - i\varepsilon t_\alpha{}^\gamma \Sigma_\gamma{}^\beta .$$

The use of upper and lower indices is motivated by the invariance of traces and Kronecker symbols:

$$\delta t_\alpha{}^\alpha = \delta t_{\dot\alpha}{}^{\dot\alpha} = 0; \qquad \delta\delta^\beta{}_\alpha = \delta\delta^{\dot\beta}{}_{\dot\alpha} = 0 .$$

(A.7)

Compact groups can have unitary representations, for which $r^{-1\dagger} = r$ and $r^{-1T} = r^*$, so that there is no need for dotted indices: $u_\alpha = u^{\dot\alpha}$ and $u^\alpha = u_{\dot\alpha}$. *Real groups* have real representations, for which $r = r^*$ and $r^{-1\dagger} = r^T$, and again there is no need for dotted indices: $u_\alpha = u_{\dot\alpha}$ and $u^\alpha = u^{\dot\alpha}$.

The *Levi–Civita tensors* are totally antisymmetric numerical tensors of rank $n = \dim(r)$,

$$\varepsilon_{\alpha_1\cdots\alpha_n}; \qquad \varepsilon_{\dot\alpha_1\cdots\dot\alpha_n}; \qquad \varepsilon^{\alpha_1\cdots\alpha_n}; \qquad \varepsilon^{\dot\alpha_1\cdots\dot\alpha_n} ,$$

which we normalise so that

$$\varepsilon_{\alpha_1\cdots\alpha_n}\varepsilon^{\beta_1\cdots\beta_n} = \pm n! \, \delta^{\beta_1}_{[\alpha_1}\cdots\delta^{\beta_n}_{\alpha_n]}$$

(A.8)

with the sign depending on other conventions used (it is minus in the case of eqs. (A.2)) and the symbol

$[\cdots]$ denoting total antisymmetrisation normalised such that

$$X_{[[\alpha_1\cdots\alpha_n]]} = X_{[\alpha_1\cdots\alpha_n]}.$$

$$(A.9)$$

Total symmetrisation is defined accordingly and denoted by $(\cdots)$.

The transformation laws for the Levi–Civita tensors are compatible with eq. (A.8):

$$
\begin{aligned}
(r(g)\varepsilon)_{\alpha_1\cdots\alpha_n} &= \det r\,\varepsilon_{\alpha_1\cdots\alpha_n}\\
(r^{-1\mathrm{T}}(g)\varepsilon)^{\alpha_1\cdots\alpha_n} &= (\det r)^{-1}\varepsilon^{\alpha_1\cdots\alpha_n}\\
(r^*(g)\varepsilon)_{\dot\alpha_1\cdots\dot\alpha_n} &= (\det r)^*\varepsilon_{\dot\alpha_1\cdots\dot\alpha_n}\\
(r^{-1\dagger}(g)\varepsilon)^{\dot\alpha_1\cdots\dot\alpha_n} &= (\det r)^{-1*}\varepsilon^{\dot\alpha_1\cdots\dot\alpha_n}
\end{aligned}
$$

$$(A.10)$$

and if the group is unimodular ($\det r = 1;\ \mathrm{tr}\,\Sigma = 0$), the ε-tensors are numerical invariants which we normalise to plus or minus one (whichever is the more convenient) if all indices are different.

A.3. Representations of the Lorentz group

The algebra of the Lorentz group, eq. (2.18c), is normally written in terms of six Hermitian generators, arranged into an antisymmetric second-rank tensor $M_{\mu\nu}$. The algebra can, however, also be written in terms of six non-Hermitian generators $J_\pm$ with

$$J_\pm^1 \equiv \tfrac{1}{2}(M^{23} \pm \mathrm{i}M^{01}) \qquad \text{and cyclic } (1\to2\to3\to1)$$

$$(A.11)$$

and then takes the form

$$
\begin{aligned}
[J_\pm^1, J_\pm^2] &= \mathrm{i}J_\pm^3 \qquad \text{and cyclic},\\
[J_\pm, J_\mp] &= 0.
\end{aligned}
$$

$$(A.12)$$

Finite dimensional *irreducible representations* are classified by pairs of half-integer numbers (j_+, j_-) which are taken from the eigenvalues $j_\pm(j_\pm + 1)$ of the two Casimir operators $(J_\pm)^2$. The algebra is not that of SU(2) × SU(2) because the J's are not Hermitian:

$$(J_\pm)^\dagger = J_\mp.$$

$$(A.13)$$

Correspondingly, whereas the J's can be represented by finite dimensional Hermitian matrices, the $M_{\mu\nu}$'s cannot. The representation $(\tfrac{1}{2}, 0)$, e.g., has

$$r(J_+) = \tfrac{1}{2}\sigma; \qquad r(J_-) = 0$$

$$(A.14)$$

(σ = Pauli matrices) and hence

$$r(M^{0i}) \equiv \tfrac{1}{2}\sigma^{0i} = -\tfrac{1}{2}\mathrm{i}\sigma^i; \qquad r(M^{12}) \equiv \tfrac{1}{2}\sigma^{12} = \tfrac{1}{2}\sigma^3 \qquad \text{and cyclic}.$$

$$(A.15)$$

The group generated by these matrices is SL(2, c). For the representation $(0, \frac{1}{2})$ we have

$$r(J_+) = 0; \qquad r(J_-) = \tfrac{1}{2}\sigma \tag{A.16}$$

and hence

$$r(M^{0i}) \equiv \tfrac{1}{2}\bar{\sigma}^{0i} = \tfrac{1}{2}i\sigma^i; \qquad r(M^{12}) \equiv \tfrac{1}{2}\bar{\sigma}^{12} = \tfrac{1}{2}\sigma^3 \quad \text{and cyclic}. \tag{A.17}$$

We see that in either case the generators L of the proper rotations are represented by $\tfrac{1}{2}\sigma$.

Since $\bar{\sigma}^{\mu\nu} = (\sigma^{\mu\nu})^\dagger$, it is appropriate to assign "index types" $(\sigma^{\mu\nu})_\alpha{}^\beta$ and $(\bar{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$ to these representations. We also have the *self-duality* relations

$$\sigma_{\mu\nu} = \tfrac{1}{2}i\varepsilon_{\mu\nu\kappa\lambda}\sigma^{\kappa\lambda}; \qquad \bar{\sigma}_{\mu\nu} = -\tfrac{1}{2}i\varepsilon_{\mu\nu\kappa\lambda}\bar{\sigma}^{\kappa\lambda} \tag{A.18}$$

which reflect the fact that there are only three rather than six linearly independent traceless 2×2 matrices.

A.4. Two-spinor notation

The two-spinor indices which correspond to these finite dimensional representations of the Lorentz group can be raised and lowered by means of the two-dimensional Levi–Civita tensors which we normalise in the following way:

$$\varepsilon_{12} = \varepsilon^{12} = -\varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{\dot{1}\dot{2}} = +1. \tag{A.19}$$

If we define the *raising of spinor indices* by

$$\psi^\alpha \equiv \varepsilon^{\alpha\beta}\psi_\beta; \qquad \bar{\psi}^{\dot\alpha} \equiv \bar{\psi}_{\dot\beta}\varepsilon^{\dot\beta\dot\alpha}, \tag{A.20}$$

then the lowering is

$$\psi_\alpha = \psi^\beta\varepsilon_{\beta\alpha}; \qquad \bar{\psi}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar{\psi}^{\dot\beta}. \tag{A.21}$$

There is a wide variety of different conventions in use for (A.19–21), with various advantages and disadvantages. The present notation has the advantage that reality relationships and ε-tensors are covariant,

$$(\psi^\alpha)^* = \bar{\psi}^{\dot\alpha} \iff (\psi_\alpha)^* = \bar{\psi}_{\dot\alpha}. \tag{A.22}$$

$$\begin{aligned} \varepsilon^{\alpha\beta} &= \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\varepsilon_{\gamma\delta} \\ \varepsilon^{\dot\alpha\dot\beta} &= \varepsilon_{\dot\gamma\dot\delta}\varepsilon^{\dot\gamma\dot\alpha}\varepsilon^{\dot\delta\dot\beta}, \end{aligned} \tag{A.23}$$

and the disadvantage that the mixed-index ε-tensors are antisymmetric,

$$\varepsilon_\alpha{}^\beta = -\varepsilon^\beta{}_\alpha = \delta_\alpha^\beta; \qquad \varepsilon^{\dot\alpha}{}_{\dot\beta} = -\varepsilon_{\dot\beta}{}^{\dot\alpha} = \delta_{\dot\beta}^{\dot\alpha}, \tag{A.24}$$

and that $(\varepsilon_{\alpha\beta})^* = -\varepsilon_{\dot\alpha\dot\beta}$.

As a general rule, unwritten spinor indices are *contracted* from "ten to four" if they are undotted and from "eight to two" if they are dotted,

$$\zeta\psi \equiv \zeta^\alpha\psi_\alpha = -\zeta_\alpha\psi^\alpha; \qquad \bar\psi\bar\zeta \equiv \bar\psi_{\dot\alpha}\bar\zeta^{\dot\alpha} = -\bar\psi^{\dot\alpha}\bar\zeta_{\dot\alpha}. \tag{A.25}$$

We define *complex conjugation* to include reversal of the order of spinors, and get

$$(\zeta\psi)^* = (\zeta^\alpha\psi_\alpha)^* = (\psi_\alpha)^* (\zeta^\alpha)^* = \bar\psi_{\dot\alpha}\bar\zeta^{\dot\alpha} = \bar\psi\bar\zeta. \tag{A.26}$$

Since the ε's are antisymmetric, we must expect and indeed find that the matrices $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ are symmetric for every index position:

$$(\sigma^{\mu\nu})_{\alpha\beta} = (\sigma^{\mu\nu})_{\beta\alpha}; \qquad (\sigma^{\mu\nu})_\alpha{}^\beta = (\sigma^{\mu\nu})^\beta{}_\alpha, \text{ etc.} \tag{A.27}$$

The four matrices $(\sigma^\mu)_{\alpha\dot\beta}$ with $\sigma^0 = \mathbb{1}$ and $\sigma^i =$ Pauli-matrices,

$$\sigma^\mu = (\mathbb{1}, \boldsymbol\sigma), \tag{A.28}$$

are, up to a factor, the only solution of the equation

$$0 = \tfrac{1}{2}\sigma^{\kappa\lambda}\sigma^\mu - \tfrac{1}{2}\sigma^\mu\bar\sigma^{\kappa\lambda} - i(\sigma^\kappa\eta^{\lambda\mu} - \sigma^\lambda\eta^{\kappa\mu}) \tag{A.29}$$

which says that $\sigma^\mu{}_{\alpha\dot\beta}$ is a numerically invariant tensor if the index μ is taken to transform according to the vector representation of $O(1, 3)$. The σ^μ are the *Clebsch–Gordan coefficients* which relate the $(\tfrac{1}{2}, \tfrac{1}{2})$ of $SL(2, c)$ to the vector of $O(1, 3)$. Accordingly, we can associate a mixed bi-spinor with every vector:

$$\not b_{\alpha\dot\beta} \equiv b_\mu(\sigma^\mu)_{\alpha\dot\beta}. \tag{A.30}$$

We can also define matrices $\bar\sigma^\mu$ by

$$(\bar\sigma^\mu)^{\dot\alpha\beta} \equiv (\sigma^\mu)^{\beta\dot\alpha} \tag{A.31}$$

and find that $\bar\sigma^\mu = (\mathbb{1}, -\boldsymbol\sigma)$ and

$$b_\mu = \tfrac{1}{2}(\not b\bar\sigma_\mu)_\alpha{}^\alpha = \tfrac{1}{2}(\not b\sigma_\mu)^{\dot\alpha}{}_{\dot\alpha} \qquad \text{with} \qquad \bar{\not b}^{\dot\alpha\beta} \equiv b_\mu(\bar\sigma^\mu)^{\dot\alpha\beta}. \tag{A.32}$$

Finally, the various σ-matrices are related by the properties

$$\sigma^\mu\bar\sigma^\nu = \eta^{\mu\nu} - i\sigma^{\mu\nu}$$

$$\bar\sigma^\mu\sigma^\nu = \eta^{\mu\nu} - i\bar\sigma^{\mu\nu}. \tag{A.33}$$

A.5. Four-spinor notation; Majorana spinors

Dirac matrices can be constructed from the 2×2 matrices σ^μ and $\bar{\sigma}^\mu$ in the following way:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{and} \quad \sigma^{\mu\nu} \equiv \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}. \tag{A.34}$$

These fulfill

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \text{and} \quad \tfrac{1}{2}\mathrm{i}[\gamma^\mu, \gamma^\nu] = \sigma^{\mu\nu} \tag{A.35}$$

and act naturally on four-spinors which are composed from a chiral and an anti-chiral two-spinor:

$$\psi \equiv \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}. \tag{A.36}$$

The adjoint spinor is

$$\bar{\psi} \equiv (\lambda^\alpha, \bar{\chi}_{\dot{\alpha}}), \tag{A.37}$$

and the charge-conjugate spinor is

$$\psi^c \equiv \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}; \qquad \overline{\psi^c} \equiv (\psi^c)^\dagger A = (\chi^\alpha, \bar{\lambda}_{\dot{\alpha}}). \tag{A.38}$$

These spinors are related through 4×4 matrices

$$A \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad C \equiv \begin{pmatrix} -\varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}; \qquad C^{-1} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{A.39}$$

in the following way

$$\bar{\psi} = \psi^\dagger A; \qquad \psi^c = C\bar{\psi}^\mathrm{T}. \tag{A.40}$$

The matrices A and C intertwine the representation γ_μ of the Dirac algebra with the *equivalent representations* $\gamma_\mu^\dagger$ and $-\gamma_\mu^\mathrm{T}$, respectively:

$$A\gamma_\mu A^{-1} = \gamma_\mu^\dagger; \qquad C^{-1}\gamma_\mu C = -\gamma_\mu^\mathrm{T}. \tag{A.41}$$

It follows that the 16 linearly independent matrices A, $A\gamma_\mu$, $A\sigma_{\mu\nu}$, $\mathrm{i}A\gamma_\mu\gamma_5$ and $A\gamma_5$ are Hermitian, the six matrices C, $\gamma_\mu\gamma_5 C$ and $\gamma_5 C$ are antisymmetric, and the ten matrices $\gamma_\mu C$ and $\sigma_{\mu\nu}C$ are symmetric. Definition and properties of γ_5 are

$$\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3; \qquad (\gamma_5)^2 = -1; \qquad \gamma_5 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \tag{A.42}$$

and the projection operators $\frac{1}{2}(\mathbb{1} \pm i\gamma_5)$ project out the chiral components χ_α and $\bar{\lambda}^{\dot\alpha}$ of ψ.

In general, the two chiral components of a Dirac spinor are unrelated. If, however, there is a *Majorana condition*

$$\psi = \psi^c ,\qquad\qquad\qquad\qquad (A.43)$$

i.e. if

$$\bar{\lambda}_{\dot\alpha} = (\chi_\alpha)^\dagger \quad\text{and}\quad \psi = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix},\qquad\qquad (A.44)$$

then we have a *Majorana spinor* with only half as many independent components.

There is a particular representation of the γ-matrices, unitarily equivalent to the one defined above, for which $-C = A = \gamma_0$. Then all γ_μ are purely imaginary (for our metric) and a Majorana spinor has real components. This representation is called the "Majorana representation", and should not be confused with the concept of a Majorana spinor: eqs. (A.40) and the condition (A.43) are representation independent. They imply a relationship between the components of ψ and $\bar{\psi}$ which only in a particular representation means that the components are actually real.

Majorana spinors can only exist because $\psi^{cc} = +\psi$ for spinors in four-dimensional Minkowski space–time.

A.6. Properties of bi-spinors; Fierz formula

For two spinors ζ and ψ whose components anticommute, we have the following relations:

$$\bar{\zeta}\psi = \bar{\psi}^c\zeta^c ; \qquad \bar{\zeta}\gamma_\mu\gamma_5\psi = \bar{\psi}^c\gamma_\mu\gamma_5\zeta^c; \qquad \bar{\zeta}\gamma_5\psi = \bar{\psi}^c\gamma_5\zeta^c$$

$$\bar{\zeta}\gamma_\mu\psi = -\bar{\psi}^c\gamma_\mu\zeta^c; \qquad \bar{\zeta}\sigma_{\mu\nu}\psi = -\bar{\psi}^c\sigma_{\mu\nu}\zeta^c .\qquad (A.45)$$

The completeness of the 16 Dirac matrices in 4×4 space can be used to derive the following *Fierz rearrangement* formula for two anticommuting spinors:

$$\psi\bar{\zeta} = -\tfrac{1}{4}(\bar{\zeta}\psi) - \tfrac{1}{4}\gamma_\mu(\bar{\zeta}\gamma^\mu\psi) - \tfrac{1}{8}\sigma_{\mu\nu}(\bar{\zeta}\sigma^{\mu\nu}\psi) - \tfrac{1}{4}\gamma_\mu\gamma_5(\bar{\zeta}\gamma^\mu\gamma_5\psi) + \tfrac{1}{4}\gamma_5(\bar{\zeta}\gamma_5\psi) .\qquad (A.46)$$

Together with properties of γ-matrices, this gives us

$$\psi\bar{\psi} - \gamma_5\psi\bar{\psi}\gamma_5 = -\tfrac{1}{2}(\bar{\psi}\psi) + \tfrac{1}{2}\gamma_5(\bar{\psi}\gamma_5\psi) - \tfrac{1}{4}\sigma_{\mu\nu}(\bar{\psi}\sigma^{\mu\nu}\psi) .\qquad (A.47)$$

For a Majorana spinor ψ, the last term vanishes because of (A.45), and multiplying with ψ from the right, we find that

$$\psi(\bar{\psi}\psi) = \gamma_5\psi(\bar{\psi}\gamma_5\psi) .\qquad\qquad (A.48)$$

A.7. Dirac matrices for arbitrary space–times

Dirac matrices Γ_a, $a = 1, \ldots d$, are defined to be irreducible representations of the algebra

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \cdot 1 \tag{A.49}$$

for arbitrary dimensions d and arbitrary signatures of the metric $\eta_{ab} = \pm\delta_{ab}$. We denote by $d_\pm$ the number of plus and minus signs in the metric:

$$d = d_+ + d_- . \tag{A.50}$$

Theorem I (without proof):

For a given *even* dimension and a given signature of the metric, all irreducible representations of the Dirac algebra (A.49) are *equivalent* and are $n \times n$ matrices with

$$n = 2^{d/2} . \tag{A.51}$$

This means that for any two representations $\{\Gamma_a\}$ and $\{\Gamma'_a\}$, there is a non-singular matrix S such that

$$\Gamma'_a = S\Gamma_a S^{-1} . \tag{A.52}$$

Theorem II:

For a given *odd* dimension and a given signature, there are *two equivalence classes* of irreducible representations in terms of $n \times n$ matrices with

$$n = 2^{(d-1)/2} . \tag{A.51'}$$

If $\{\Gamma_a\}$ is in one equivalence class then $\{-\Gamma_a\}$ is in the other.

I give a proof only for the last statement: clearly, $\{-\Gamma_a\}$ represents (A.49) if $\{\Gamma_a\}$ does. For all dimensions we can define a "generalised γ_5" as

$$\Gamma_{d+1} \equiv \Gamma_1 \cdots \Gamma_d . \tag{A.53}$$

As a consequence of eq. (A.49), Γ_{d+1} will *anticommute* with all Γ_a for *even* d, but *commute* for *odd* dimensions where it is therefore a multiple of the unit matrix (Schur's lemma). If there were an S such that

$$S\Gamma_a S^{-1} = -\Gamma_a , \tag{A.54}$$

then

$$S\Gamma_{d+1} S^{-1} = (-1)^d \Gamma_{d+1} , \tag{A.55}$$

which for odd d is in contradiction with $\Gamma_{d+1} \propto 1$. For even d we have indeed

$$\Gamma_{d+1}\Gamma_a\Gamma_{d+1}^{-1} = -\Gamma_a \qquad \text{(even } d) \tag{A.56}$$

as an intertwiner of $\{\Gamma_a\}$ and $\{-\Gamma_a\}$.

Equivalence for even dimensions:

All of the following are representations of (A.49):

$$\Gamma_a, -\Gamma_a, \Gamma_a^\dagger, -\Gamma_a^\dagger, \Gamma_a^T, -\Gamma_a^T, \Gamma_a^*, -\Gamma_a^* .$$ (A.57)

We introduce intertwiners

$$A\Gamma_a A^{-1} = \Gamma_a^\dagger$$ (A.58)

$$C^{-1}\Gamma_a C = -\Gamma_a^T$$ (A.59)

which together with Γ_{d+1} allow us to transform any two of the representations (A.57) into each other. Thus, e.g.,

$$D^{-1}\Gamma_a D = -\Gamma_a^*$$ (A.60a)

with

$$D = CA^T .$$ (A.60b)

For any two representations, the matrix S which intertwines them is unique up to a numerical factor (this is a stronger version of Schur's lemma). From the Hermitian adjoint of (A.58), the negative of (A.56), the transposed of (A.59) and the complex conjugate of (A.60) we get thus

$$A = \alpha A^\dagger; \qquad \Gamma_{d+1} = \beta\Gamma_{d+1}^{-1}; \qquad C = \eta C^T; \qquad D = \delta D^{-1*} .$$ (A.61)

These equations are consistent only if the numbers α, η and δ satisfy

$$\alpha\alpha^* = \eta^2 = 1 \qquad \text{and} \qquad \delta = \delta^* .$$ (A.62a)

A free phase and a free scale factor in the definition of A can be used to fix

$$\alpha = |\delta| = 1$$ (A.62b)

(the only redefinitions still possible are then $C \to cC$; $A \to |c|^{-1}A$). The remaining quantities (β and the signs of η and of δ) are invariants that cannot be tampered with. They are defined by the metric, i.e. by d_+ and d_-, and are given in tables A.2 to A.4 below.

Under an arbitrary similarity transformation (A.52), the matrices A, C and D transform as

$$A' = S^{-1\dagger}AS^{-1}; \qquad C' = SCS^T; \qquad D' = SDS^{-1*}$$ (A.63)

(there is one free numerical factor in this, corresponding to the rescaling which is still possible for C; it has been set to one). Equations (A.63) and (A.55) are consistent with (A.61).

For the matrix

$$\tilde{D} \equiv \Gamma_{d+1}^{-1} D \qquad\qquad (A.64)$$

we have

$$\tilde{D}^{-1}\Gamma_a\tilde{D} = \Gamma_a^* \qquad\qquad (A.65)$$

and

$$\tilde{D} = \delta\tilde{D}^{-1*} \quad \text{with} \quad \bar{\delta} = \beta\delta . \qquad\qquad (A.66)$$

Equivalence classes for odd dimensions:

For odd d, the question which of the eight representations (A.57) fall into which equivalence class is determined by the behaviour of the product Γ_{d+1} under the respective transformations. With

$$A\Gamma_a A^{-1} = \pm\Gamma_a^\dagger; \qquad C^{-1}\Gamma_a C = \pm\Gamma_a^{\mathrm{T}} \qquad\qquad (A.67)$$

Table A.1
The equivalence classes which contain Γ_a for arbitrary odd dimensions

	$d = 1, 5, 9 \bmod 4$	$d = 3, 7, 11 \bmod 4$
d_- even	$\Gamma_a, \Gamma_a^\dagger, \Gamma_a^{\mathrm{T}}, \Gamma_a^*$	$\Gamma_a, \Gamma_a^\dagger, -\Gamma_a^{\mathrm{T}}, -\Gamma_a^*$
d_- odd	$\Gamma_a, -\Gamma_a^\dagger, \Gamma_a^{\mathrm{T}}, -\Gamma_a^*$	$\Gamma_a, -\Gamma_a^\dagger, -\Gamma_a^{\mathrm{T}}, \Gamma_a^*$

Table A.2
The value of $\beta = \Gamma_{d+1}^2$

	$d = 1, 4, 5, 8, 9 \bmod 4$	$d = 2, 3, 6, 7, 10, 11 \bmod 4$
d_- even	$+1$	-1
d_- odd	-1	$+1$

Table A.3
The sign of η (symmetry property of C). The table continues modulo 4 in d

$d = 1$	2	3	4	5	6	7	8	9	10	11	12
+	−	−	−	−	+	+	+	+	−	−	−

Table A.4
The signs of δ and $\delta\bar{\delta}$ (possible Majorana conditions, see subsection 14.1). The table continues modulo 8 in d and modulo 4 in d_-

$d =$	1	2	3	4	5	6	7	8	9	10	11	12
$d_- = 0$	+	− +	−	− −	−	+ −	+	+ +	+	− +	−	− −
1		+ +	+	− +	−	− −	−	+ −	+	+ +	+	− +
2			+	+ +	+	− +	−	− −	−	+ −	+	+ +
3				+ −	+	+ +	+	− +	−	− −	−	+ −
4					−	+ −	+	+ +	+	− +	−	− −
5						− −	−	+ −	+	+ +	+	− +
6							−	− −	−	+ −	+	+ +
7								− +	−	− −	−	+ −
8									+	− +	−	− −
9										+ +	+	− +
10											+	+ +
11												+ −

and always $A\Gamma_{d+1}A^{-1} = C^{-1}\Gamma_{d+1}C = \Gamma_{d+1} \propto 1$, we find table A.1 as result. Otherwise the analysis is exactly parallel to the case of even d, and I have given the tables A.2, A.3 and A.4, respectively, for β, η and δ. The last two tables are most easily calculated in a specific representation of the Dirac matrices.

Antisymmetric products:

For even d, all antisymmetric products

$$\Gamma_a \equiv \Gamma_a$$

$$\Gamma_{ab} \equiv \Gamma_{[a}\Gamma_{b]}$$

$$\Gamma_{abc} \equiv \Gamma_{[a}\Gamma_b\Gamma_{c]} \tag{A.68}$$

$$\vdots$$

$$\Gamma_{a_1\cdots a_d} \equiv \Gamma_{[a_1}\Gamma_{a_2}\cdots\Gamma_{a_d]}$$

are *trace orthogonal* to each other and to the unit matrix (i.e. they are traceless). Their number is

$$\sum_{k=1}^{d} \binom{d}{k} = 2^d - 1 ,$$

so that together with the unit matrix they are a *complete basis* in the space of Dirac matrices.

For all dimensions, the relationship

$$\Gamma_{a_1\cdots a_k} = \alpha\varepsilon_{a_1\cdots a_d}\Gamma^{a_{k+1}\cdots a_d}\Gamma_{d+1} \tag{A.69a}$$

holds with

$$\alpha = \frac{1}{(d-k)!}(-1)^{k(k-1)/2+d(d-1)/2} . \tag{A.69b}$$

For odd dimensions, where Γ_{d+1} is a number, this implies that only the antisymmetric products up to rank $\frac{1}{2}(d-1)$ are linearly independent. Their number is

$$\sum_{k=1}^{(d-1)/2} \binom{d}{k} = 2^{d-1} - 1 ,$$

and together with the unit matrix they, too, are complete.

It is easy, for each case at hand, to derive Hermiticity properties of $A\Gamma_{\ldots}$ and symmetry properties of $\Gamma_{\ldots}C$, as was done for $d = 4$ and $d_- = 3$ in the sentence following eq. (A.41).

The following formulas may be found useful

$$\Gamma^{a_1\cdots a_k}\Gamma_{a_1\cdots a_k} = (-1)^{k(k-1)/2} d!/(d-k)!$$

$$\Gamma^b\Gamma_{a_1\cdots a_k}\Gamma_b = (-1)^k(d-2k)\Gamma_{a_1\cdots a_k}$$

$$\Gamma^{b_1\cdots b_k}\Gamma_a\Gamma_{b_1\cdots b_k} = (-1)^{k(k-1)/2}(d-2k)\frac{(d-1)!}{(d-k)!}\Gamma_a \tag{A.70}$$

$$\Gamma^{cd}\Gamma_{ab}\Gamma_{cd} = -(d^2 - 9d + 16)\Gamma_{ab} .$$

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